QUESTIONS

IN

MATHEMATICS

The University of Toronto.

From the Author.

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MATHEMATICS

BY

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AUTHOR OF "THE CULMINATION OF THE SCIENCE OF LOGIC."

PUBLISHED BY

HERBERT C. SMITH,

16 Court St., Brooklyn, N. Y.

Electrotyped by R. HARMER SMITH & SONS, 82 Beekman St., New York.

PREFACE.

THIS book is an outcome of one recently published, entitled "The Culmination of the Science of Logic."

The striking analogy between the necessary forms of the process of reasoning and the simplest forms of geometry, exhibited by the author in that book, led him to the reflection that perhaps the processes of geometry would have been greatly simplified, and its operations therefore more easily performed, if the regular triangle and tetrahedron, instead of the square and cube, had been adopted as units of measure of surface and solidity.

The determination of this question, as the author was well aware, required a better acquaintance with such operations and processes than he possessed, he being but a tyro, in the secondary sense of that word, in mathematical science. A tyro may, however, ask questions; but to ask a question without at the same time giving some reasons for the asking, would be to obtain for it but slight and insufficient attention.

The author thereupon commenced an investigation of the subject for the purpose of finding such reasons, if any there were, other than those which had led him up to the question, intending to submit the question, with the reasons, to those qualified to consider and determine it, through the columns of some scientific journal. But the field of investigation widened as he advanced, and further questions suggested themselves, until finally the results, after many prunings down, took shape as set forth in the following pages. Some of such results may, perhaps, have no relevancy to the main question or any other of the propounded questions, but they are given because they may possibly be of service in further investigation of the subject, if it be deemed worthy of pursuit.

The main question, as before indicated, in so far, at least, as relates to the substitution of the triangle for the square, must have suggested itself to the mind of almost every thoughtful student of the science. The author himself (who, however, can hardly be said ever to have been a *student* of the science) observed and considered it in years long gone by, but with reference to the triangle only. Such consideration was necessarily very far from thorough, and resulted in his inability to see that any advantage would be gained by the substitution. Perhaps the consideration by others may have been in like manner limited and, therefore, insufficient and attended with the like result.

It may be, however, that the question has been thoroughly considered with reference to both the triangle and the tetrahedron, and if so, then the conclusion arrived at must have been adverse, or that no advantage would result from the substitution. If such be the case, the fact has never come to the knowledge of the author. The subject matter and the results of his investigation are all new to him.

But in either case he feels confident that the subject has never been considered in the light of the analogy referred to, and such analogy, if in form only, seems to him to give sufficient importance to the question to call for an attentive and exhaustive consideration in the first case, or a reconsideration in the second.

The author puts forth the book with great diffidence, but is impelled by a sense of duty. It will be manifest from the tone of his questions what his opinion in respect to each is, but such opinions—except in so far as they are supported by processes shown, whereby the operations of evolution and involution to the third degree may be readily performed—are founded only upon intuition (in the literal sense of that word), and have not the strength of convictions. If they shall be confirmed by competent authority, then a benefit will have been conferred upon mankind, which, but for the publication of the book, would, perhaps, have forever remained unknown.

If, however, on the other hand, they shall not be confirmed, then it would seem that nature, while conforming her processes in the two sciences to each other on the faces of her simplest regular figures in a most wonderful order and symmetry, has not designed the lines of such figures as her chosen paths of investigation and reasoning in mathematical science, but has rather made choice of devious paths along the lines of complex, although still regular, figures by which such investigation and reasoning can be more advantageously pursued, thus making order and symmetry merely formal, and of no significance, value or effect.

The treatise is divided into two parts, the main part and an appendix. The appendix pertains, perhaps, more to the science of Logic than to that of Mathematics. It will be found to be illustrative and fully corroborative of the doctrine of sorites as unfolded in the author's first book, "The Culmination of the Science of Logic," and in this aspect is properly an appendix. But it is also illustrative of the relations of the parts of certain geometrical figures described in the main part of the treatise, in respect as well to their construction, or rather the combinations of their parts, as to their analogy to compound processes of reasoning, and in this respect it is supplementary and entitled to be regarded as a part of the treatise.

The author can hardly indulge the feeling of assurance that he has made no mistakes, but he is confident that, if any are found, they will not be serious nor such as to affect unfavorably the general design of the work. He makes no apology for the diffuseness of his style, or the profuseness of his illustrations. He has confessed himself to be a tyro, and from such could not be expected the conciseness and precision of an expert.

If the presentation of the subject be such as to engage attention and draw forth answers to the questions, then, whatever such answers may be, the object of the author will have been accomplished.

BROOKLYN, *December 4, 1889.*

CORRIGENDA.

Page 28, 3d line. Insert "produced" after "altitude".

31, 15th line. Insert "but produced," after "triangles" and "produced" after "altitude".

35, 5th and 7th lines. Insert "combined" before "sorites" in each line.

41, 2d line from bottom. Insert "each" after "will" and strike out "adjacent".

72, 5th line. After "ber," insert "when the number of such edge or root does not exceed 10005,".

94, 3d line. After "number," insert "when the number of such root does not exceed 10005,".

130. Strike out 2d and 3d ¶¶.

132. Strike out 2d ¶. 11th line from bottom. Insert "but" before "with".

133, 9th line. After "found" insert "by investigation (as well as by reasoning)".

134. On folded sheet following, mark symbols "N" designating two points in 3d horizontal line of Fig. 3*b* and "X′," two points in like line from bottom of Fig. 19*b* "to be considered as in full-faced type".

136, 10th line from bottom. Substitute "original" for "principal".

140, 10th line. For "octrahedron" read "octahedron".

last line. Substitute "original" for "principal".

153, 4th line. Strike out "each of".

154, 4th line from bottom. Insert "exterior" before "edge".

SEE POSTSCRIPT.

QUESTIONS

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MATHEMATICS.

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§ 1. The square and the cube have served from time immemorial as units of measure of surface and solidity. The square was undoubtedly adopted, when the occasion for its use first arose (probably for the measurement of land, and hence the name geometry), because of the uniformity of its parts and its apparent simplicity: and the cube, when the occasion for its use first arose, was naturally adopted for the same qualities and as being modeled upon the square. That they are the best adapted of all geometrical figures to serve as such units of measure for all the ordinary practical purposes of every-day life, there can be no question. They have the appearance, at first sight, of being simple figures, and are very readily comprehended. But they are, in fact, both complex figures, and it would seem, that when the study of geometry came on to be pursued either for its own sake, independently of its practical application, or for that of the higher purposes to which it is applicable, and especially when questions involving the third dimension of space came under consideration, the fact that the units of measure theretofore in use for ordinary purposes only, were not the simple forms suggested by nature, would have been recognized, and the question considered, whether they should not be discarded for such higher purposes, and the two and only simple and regular forms of plane surfaces bounded by straight lines, and solids with plane surfaces (the regular triangle and regular tetrahedron), adopted in their stead, not only as actually in accord with nature, but also as being more likely to lead to simplicity in the operations and processes to be founded upon them, than the square and cube in the operations and processes founded upon them.

Whether they would or not lead to such greater simplicity is the main question, which it is the object of this book to submit to those qualified to consider and determine it.

That two units of measure of space may be concurrently in use, the one apparently simple but in fact complex and employed for the ordinary purposes of life, and the other the true, simple unit in actual accord with nature and employed for higher purposes, is evidenced by the use of two different units of measure of time, viz.: the solar day corresponding to the former, and the sidereal day corresponding to the latter.

§ 2. The square in its simplest analysis is composed of two equal right-angled isosceles triangles. It is, in fact, if the expression may be allowed, a double unit, consisting of two right-angled triangular units combined. It alone, or in conjunction with the linear unit upon which it is described, can be applied only to the measurement of areas in the form of right-angled parallelograms, all sides of which are accessible throughout their whole extents. Areas, the angles of which are not all right angles, and right-angled areas the boundaries of which are not accessible throughout their whole extents, can only be measured by means of the triangular unit. But the triangular unit, which is the half of a square, is irregular, consisting of three lines, of which two only are equal to each other, the third being incommensurable, except in power, with each of the other two, and three angles, of which also two only are equal to each other, the third being double each of the other two.

On the other hand, the regular triangle has all its lines and angles equal to each other respectively, being both equilateral and equiangular.

The hypothenuse of the right-angled isosceles triangle is, as will be hereinafter shown, the real linear unit upon which the square is constructed. By means of it as the radius of a circle and invariably representing unity, and the varying dimensions of the other two sides of all possible right-angled triangles formed upon it within the circle, are all angles measured, but only by means of squares considered as formed upon the sides of such triangles. All angles may be measured directly by the regular triangle and without recourse to squares, as will be hereinafter shown.

The diameters of the inscribed and circumscribed circles of the square are incommensurable with each other, except in power, and one only, that of the inscribed circle, is commensurable with, or rather equal to, the altitude of the square; but the diameters of the inscribed and circumscribed circles of a regular triangle are not only commensurable with each other, but both are also commensurable with, though neither equal to, the altitude of the triangle.

A square can only be increased in area and its form preserved by extending equally two of its sides having a common point, each in its direction from such point, and drawing from the extremities of such produced sides two lines parallel to its other two sides until the lines so drawn meet at a common point; but a regular triangle may be increased and its form preserved by a like extension of any two of its sides, and connecting the extremities of the produced sides by a single straight line which will be parallel to its third side.

While it is not of the very essence of a unit of measure that it should have all its similar parts equal to each other, it is, nevertheless, of the highest importance as conducive to simplicity in application and calculation that such should be the case, and the square was adopted because of its conformity to this seeming requirement. But the square is not the real unit.

Should not, therefore, the triangle, which is the real unit, be uniform in respect to all its similar parts, and instead of the irregular right-angled isosceles triangle be the regular equilateral and equiangular triangle?

§ 3. The cube is a highly complex figure, being composed in its simplest analysis of five figures, of which one only is regular, viz.: a regular tetrahedron, the nucleus of the cube and wholly hidden therein, and having for its six edges diagonals of the six faces of the cube. The remaining four figures are irregular, viz.:

equal right-angled tetrahedra, each having as its base a regular triangle equal to each of the faces of the hidden regular tetrahedron, and each superposed on one of such faces. The volume of the cube is three times that of the included regular tetrahedron, each right-angled tetrahedron being one-half the volume of such regular tetrahedron and one-sixth the volume of the cube.

The diameters of the inscribed and circumscribed spheres of the cube and regular tetrahedron are incommensurable or commensurable with each other and with the altitude of the cube and tetrahedron respectively, in like manner as those of the inscribed and circumscribed circles of the square and regular triangle, relatively to each other, and to the altitude of the square and triangle respectively.

A cube can only be increased in volume and its form preserved by superposing upon each of three of its faces having a common point a parallelopipedon of equal face, and then filling up three parallelopipeds of equal length and thickness, and after them a remaining cube, but a regular tetrahedron may be increased and its form preserved by adding uniformly to any one of its faces.

§ 4. A regular tetrahedron may, by four sections beginning in the middle of four of its edges, and made parallel to the opposite faces respectively, be divided into five figures, all of which will be regular, viz.: four equal regular tetrahedra, and the fifth and interior figure a regular octahedron. If the original figure be considered as of the edge of 2, the five figures into which it is divided will each be of the edge of 1. The octahedron is equal in volume to the four tetrahedra combined. The original figure is therefore equal in volume to eight tetrahedra of the edge of 1.

If now the original figure be considered not as actually divided, but as marked on its faces with the lines of the division, there will be four faces of the octahedron visible and four invisible. If, upon the four visible faces of the octahedron there be superposed four regular tetrahedra of the edge of 1, the resulting figure will be in the form of an eight-pointed star (which may be called an oct·astron), consisting of two equal intervolved regular tetrahedra (edge 2), to both of which the interior and now wholly hidden octahedron is common. The points are the extremities of the axes of the oct·astron, of which there are four. The oct·astron is equal in volume to twelve tetrahedra (edge 1), viz.: the octahedron equal to four, and the eight superposed upon its faces. If six of these tetrahedra, three of the original figure and three of those superposed as above described, be cut off, leaving two, the points of which are the opposite extremities of any one of the axes of the oct·astron, there will remain a figure consisting of the octahedron with two tetrahedra attached to opposite faces thereof, and equal in volume to the six detached tetrahedra. If now there be three planes passed through the octahedron in line with its edges extending from one tetrahedron to the other, and through the centre of the octahedron (but in such manner that the edges between the visible faces of the octahedron and the two tetrahedra be considered as not severed), the figure will be divided into two equal parts, each consisting of a regular tetrahedron with four irregular equal right-angled tetrahedra attached, one by one of its faces and the other three each by one of its edges. If the two parts be now considered as put together again and the axis reunited at the centre, so that it will hold the two parts relatively in the same position (the figure being considered as standing or held so that the axis shall be vertical), the six right-angled tetrahedra, attached by their edges, may be folded over (three upward and three downward) on the three visible faces of each of the two regular tetrahedra, and the resulting figure will consist of two perfect cubes connected together at the middle point of an axis running through a diagonal of each. The six detached tetrahedra being equal in volume to the octahedron and two tetrahedra so changed in form, may be considered as changed also in form to two other similar cubes, although they cannot be as simply dissected, and the parts put together in the new form. Thus the oct·astron is equal in volume to four such cubes, and each such cube is equal in volume to three tetrahedra (edge 1). Each cube has the diagonal of each of its faces equal to the edge (1) of its included tetrahedron, and each edge of the cube is therefore equal to $\sqrt{.5} = .7071$.

Does it not clearly appear from the foregoing analysis of the regular tetrahedron (the simplest form of three dimensions in nature), the building up or completion thereon, or rather upon the octahedron therein contained, of the oct·astron and the subsequent dissection of the latter for the purpose of finding its contents in the complex form of cubes, that the diagonal is the real linear unit upon which the squares, the bounding faces of the cubes, are constructed ?

§ 5. The triangle of geometry is the analogue of the syllogism of logic in respect to form, and the quadrilateral is the analogue of the simple sorites (syllogism of four terms) in either the ascending or descending direction of the process of reasoning, but limited in each case in so far as the notion of space comes under consideration to space of two dimensions. The tetrahedron is the analogue of the sorites in both the ascending and descending directions of the process of reasoning combined, ascending first from subject to predicate on two faces of the tetrahedron, and then descending from predicate (now become subject) to subject (now become predicate) on the other two faces, or *vice versa*. Such combined sorites may, however, be considered as in either direction throughout, the first progressive and the second regressive, or *vice versa*.

In the following figures, each equal to the other and each in the form of a quadrilateral composed of two regular triangles, let the full continuous lines represent the only lines which can be actually measured, and the dotted, and partly dotted lines, those which are the results of processes of reasoning, and let the arrows introduced in dotted lines indicate the directions in which the points toward which they are directed can be seen from the points from which they are directed respectively, and let the ultimate points, N in the first figure and X in the second, be regarded as inaccessible.

Fig. 1.

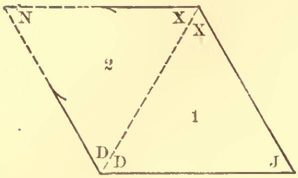
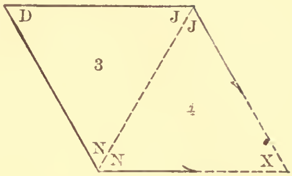


Fig. 2.



The letters by which the points of the figures are designated are the symbols adopted by the author to represent the terms of the sorites, their logical significations (as such terms were named also by the author) being as follows (reading the first column of symbols downward and the second upward in each case in connection with the logical significations):

Descending from		Logical Significations.		*Ascending to*
Subject	X	Maximus term	X	Predicate
to	J	Major-middle term	J	*from*
	D	Minor-middle term	D	
Predicate	N	Magnus term	N	Subject

Each triangle is the analogue of a syllogism; each quadrilateral the analogue of a simple sorites, viz.: that composed of triangles 1 and 2 taken together, without the diagonal, in the descending direction of the process of reasoning, and that composed of triangles 3 and 4 taken together, without the diagonal, in the ascending direction. The diagonal in each case represents the unexpressed conclusion of the first, which is the unexpressed premise of the second of a series of two syllogisms into which each sorites may be fully expanded.

The sorites are as follows, those in the descending direction being as shown in Fig. 1, and those in the ascending direction as in Fig. 2.

Progressive descending.

X comprehends J,
J comprehends D,
D comprehends N;
∴ X comprehends N.

Regressive ascending.

J is comprehended in X,
D is comprehended in J,
N is comprehended in D;
∴ N is comprehended in X.

Progressive ascending.

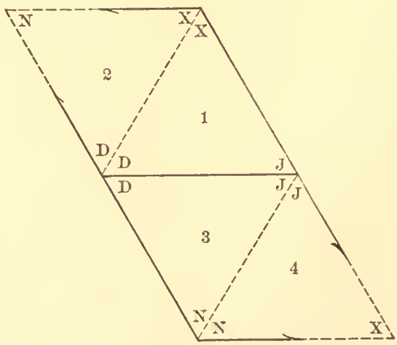
N is comprehended in D,
D is comprehended in J,
J is comprehended in X;
∴ N is comprehended in X.

Regressive descending.

D comprehends N,
J comprehends D,
X comprehends J;
∴ X comprehends N.

If now the two figures be considered as put together on their only common line capable of actual measurement, J D in the first figure and D J in the second (analogue of the middle premise of each sorites), they will present the following figure:

Fig. 3.



which may be folded on its interior lines in the form of a regular tetrahedron, the points N N, N, and X X, X, respectively, meeting. The two partly dotted lines, D N in triangle 2 and J X in triangle 4, will coincide with the continuous lines N D in triangle 3 and X J in triangle 1 respectively, and the other two partly dotted lines X N in triangle 2 and N X in triangle 4 (analogues of the ultimate conclusions of the two sorites), will coincide and form a continuous line with the arrow heads lying in adjacent faces, but pointing in opposite directions.

Thus geometry through its simplest forms makes clear to the eye as well as to the understanding that its underlying science—logic—must in its simplest forms exhibit the relations of three terms, and may extend but is limited to four, any advance beyond the fourth term being impossible except by actual investigation, such investigation in the cases represented by the figures (except considered as folded in the form of a solid figure) being also impossible. The coincidence of the lines of the figures as above described demonstrates the infallibility of the logical processes.

The square, in like manner, represents the sorites resolvable by means of its diagonal into two syllogisms, as in the following figures :

Fig. 4.

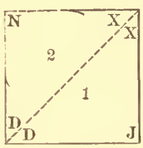
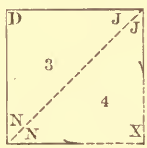
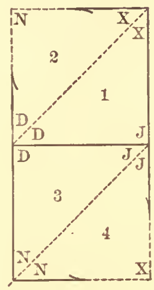


Fig. 5.



But if the two squares are put together in like manner as before, thus:

Fig. 6.



they cannot be folded on their interior lines so as to form a solid figure.

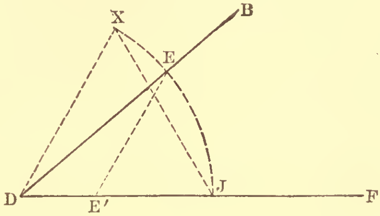
Six squares are required to form a cube and three combinations of two sorites each (corresponding in number to the three regular tetrahedra to which the cube is equal in volume) are required to describe the surface instead of one, as in the case of the tetrahedron.

Does not nature clearly point to the regular tetrahedron instead of the cube as the simplest unit of measure of solidity?

§ 6. The regular triangle as a unit of measure with an arc of a circle described on any side thereof from the opposite point as a centre, may be applied to the determination of any plane angle as follows:

Let the angle B D F, in the following figure, be the angle to be determined, and let the unit of measure, X J D, with an arc described on the side X J from the point D as a centre, be applied to it as shown.

Fig. 7.



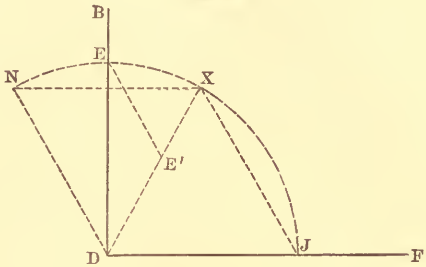
From the point E, where the line B D intersects the arc, draw E E′ parallel to X D, and measure the line so drawn. The angle E E′ D will always be an angle of 120° at whatever point in the line D J its vertex may be. Then, by means of the two known lines D E and E E′ and the known angle E E′ D opposite one of them (D E), the angle in question may be found. There will be no ambiguity. The line D E opposite the known angle E E′ D will always be greater than the line E E′.

In like manner, a line may be drawn from the point E to the line X D parallel to the line D J and measured, and by means thereof and the known line D E and the angle of 120° opposite thereto on the line X D, the angle E D X may be determined.

If the angle to be determined exceed 60° and be less than 120°, the unit of measure will have to be applied the second time, and if it exceed 120°, the third time, and so on until the circuit be completed, the line E E′ being drawn in the last application in which B D will intersect the side of the unit opposite the point D, and the sum of the preceding applications being added to that therein found.

Thus, if the angle to be determined be a right angle, the line B D will lie in the second application of the unit, as in the following figure:

Fig. 8.



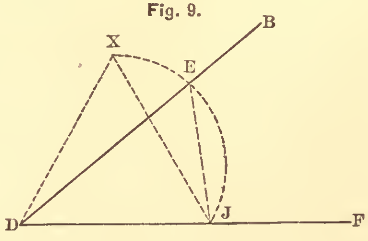
The point E′ will be in the line D X, and the angle E D E′, the same as B D X, will be found to be an angle of 30°, to be added to X D F 60°, as measured by the first application of the unit.

$$B\,D\,X\,(30°) + X\,D\,F\,(60°) = B\,D\,F\,(90°).$$

The triangular unit of measure having described on one side thereof an arc of its circumscribing circle may also be applied to the determination of plane angles as follows:

Let B D F be the angle to be determined, and let the unit with an arc of its circumscribing circle described on its side X J be applied to it, as in the following figure:

Fig. 9.



From the point E, where the line B D intersects the arc, draw E J and measure the line so drawn. The angle D E J will always be an angle of 60° at whatever point in the arc its vertex may be. Then, in like manner as before, by means of the two known lines D J and E J and the known angle D E J opposite one of them (D J), the angle in question may be ascertained. As before, there will be no ambiguity. The line D J opposite the known angle D E J will always be greater than the line E J.

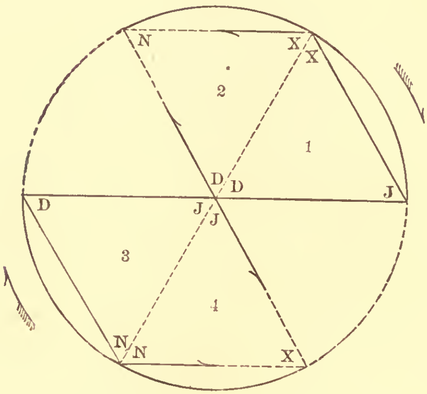
If the chord of the arc X E be drawn, the angle D E X will also be an angle of 60°, and in like manner the angle E D X may be found.

If the angle to be determined exceed 60°, then, as before, the unit of measure, if with but one arc described thereon, as in the figure, will have to be applied the second time, and if it exceed 120° the third time, and so on until the circuit be completed.

§ 7. But the circuit can never be completed except by the continued applications of the unit of measure or by independent processes of reasoning, for which further investigation will be required.

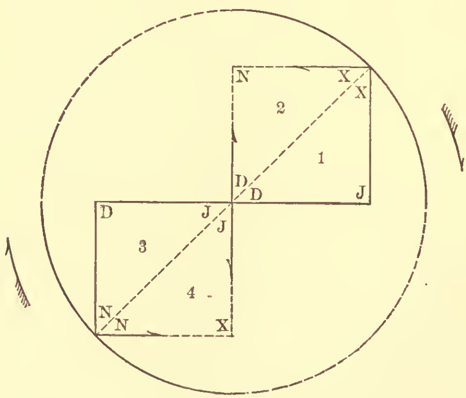
This will be manifest by the consideration of the following figure, in which the two quadrilaterals, having their points designated by the logical symbols, are brought together in such manner that from a common point designated by the symbol of the third term in each direction (minor-middle descending and major-middle ascending) a circle may be described about them in the circumference of which all the other points will lie.

Fig. 10.



Let this figure now be compared with one similarly drawn, but in which the quadrilaterals are squares, as follows :

Fig. 11.

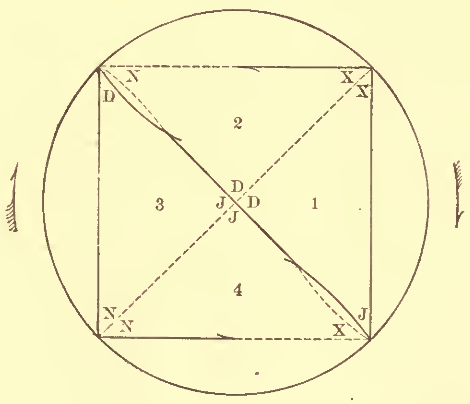


In the former figure, the points designated by the symbols of all the terms of the sorites have positions in the circumference except those of the third in each process, which are both middle terms, and designate the centre; in the latter, only the points designated by the symbols of the terms of beginning respectively have positions in the circumference with those of the same third terms designating the centre, and all the points designated by the symbols of all the other terms fall within the area of the circle.

Which of these figures gives promise, rather than the other, of simplicity in the operations and processes to be founded upon them?

§ 8. A single square having two diagonals, and with a circle circumscribed may, however, be exhibited, as in the following figure:

Fig. 12.



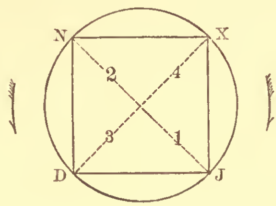
All the points, including that of intersection of the diagonals, are designated by the logical symbols, and have positions in the circumference and at the centre of the circle. In order to represent the ultimate points, N in the descending process and X in the ascending, as inaccessible, each half of one of the diagonals is exhibited in part in two curved lines, one continuous and the other dotted, the dotted lines being considered as following the arrow heads introduced.

Here the combination of figures representing the two sorites is a combination of two right-angled triangles, each consisting of two smaller triangles, 1 and 2 taken together and 3 and 4 taken together, instead of a combination of quadrilaterals, as in Figs. 10 and 11 ; but each greater triangle is, in fact, a quadrilateral, the diagonals each consisting of two lines, forming one and the same straight line.

Two of the exterior points, it will be seen, are marked each with two different symbols, viz.: one with J and X, and the other with D and N, with a line separating the symbols in each case instead of a space, as in the other figures. The line X J in triangle 1 is greater than J X in triangle 4, and N D in triangle 3 is greater than D N in triangle 2. The inequality of these lines, and generally the inequality of the lines and angles in this figure as compared, or rather contrasted, with the equality in all respects in Fig. 10, besides the confusion arising from designating two of the points each by two different symbols must, as it would seem to the author, determine the question of their likelihood respectively of leading to simplicity in favor of the latter.

Again, a square with two diagonals and a circle circumscribed may be exhibited in which the symbols of all the terms of the sorites shall appear each but once, and all designating points in the circumference of the circle, as in the following figure :

Fig. 13.



The square in this figure must be considered as consisting of two squares, one superposed on the other, and let it be considered that the one superposed is that consisting of triangles 1 and 2, in which the sorites is in the descending direction, and that the diagonal on the one beneath consisting of triangles 3 and 4, and in which the sorites is in the ascending direction, is not drawn on the one superposed, but shows through the paper.

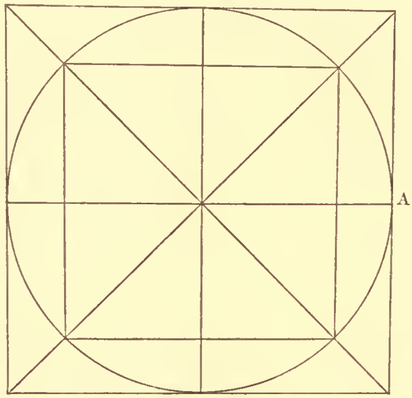
It will now be seen that the reasoning process in the ascending direction is in the reverse circular direction of that in which it has been heretofore exhibited, and that the two processes run counter to each other, as indicated by the arrows on the outside of the circle instead of in the same course, as in all previous illustrations, and as in like manner therein indicated.

The diagonals are also in opposite directions, and their apparent point of intersection is undesignated.

The objections in respect to the next preceding figure as to the inequality of certain specified lines and the designation of two points, each by two different symbols, do not apply in this case, but the general objection of inequality of the lines and angles, as contrasted with the equality of those in Fig. 10, does apply.

§ 9. The following figure exhibits a completed circuit of the square unit of measure, or rather of the right-angled triangular unit, with a circumscribed circle and square to which latter the diagonals of the inscribed square are produced.

Fig. 14.



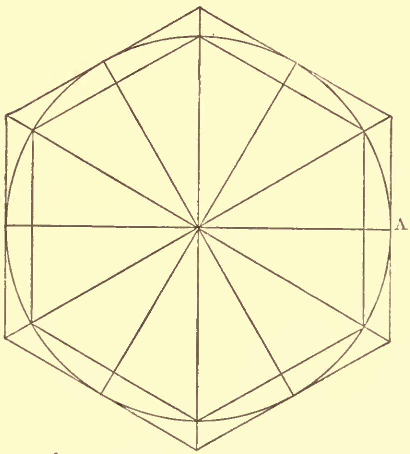
Considered as a circuit of the right-angled triangular unit of measure, it is such with reference to the right-angled triangles into which the inscribed square is divided by the diagonals, each of which has its right angle at the centre of the circle, and not with reference to the right-angled triangles into which the smaller squares are divided, none of which has its right angle at the centre, and considered with reference to the logical processes, it is founded upon Fig. 12 and not upon Fig. 13, in which the all-important point, the centre of the circle, is undesignated. The semi-diameters of the circle by which the inscribed square is divided into four smaller squares constitute lines of altitude of the right-angled triangular units produced to the circumference of the circle.

Considered as the completed circuit of the square unit of measure, it is such with reference to the smaller squares, and is founded upon Fig. 11.

This may be said to be the primary figure in which are delineated all the lines constituting the trigonometrical functions, those in the figure being the functions of an arc of 45°, or any multiple thereof by an *odd* number.

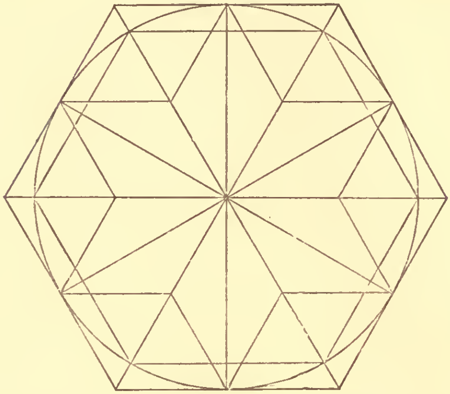
The following figure exhibits a completed circuit of the regular triangle as the unit of measure, with a circumscribed circle and hexagon, to which latter the sides of the triangles (corresponding to the half-diagonals of the inscribed square in the preceding figure) are produced, and with lines of altitude of the triangles proceeding from the centre of the circle produced to the circumference in like manner as in the preceding figure.

Fig. 15.



This figure may also be exhibited with additional lines, as follows:

Fig. 15 *a*



The horizontal diameter of the circle in this phase of the figure consists of sides (one each) of two triangles instead of lines of altitude, as in the original phase and in Fig. 14, and the added lines are those directed to be drawn in the description of the first method of applying the triangular unit of measure to the determination of plane angles hereinbefore contained. The figures described by such lines (in respect to each triangle) in connection with half the produced sides of the triangles to which they are drawn respectively, are perfect rhombs, with two angles of 60° each and two of 120° each, and the lines so drawn are equivalent to the sines and cosines, as in Fig. 14.

If the lines in this figure, in either phase, were adopted as trigonometrical functions, each would be found to have a definite relation to the arc with which it is connected, which would vary as the arc should vary similarly to the lines now in use, as in Fig. 14, and they would be the trigonometrical functions of an arc of 30°, or of any multiple thereof, by *any whole* number.

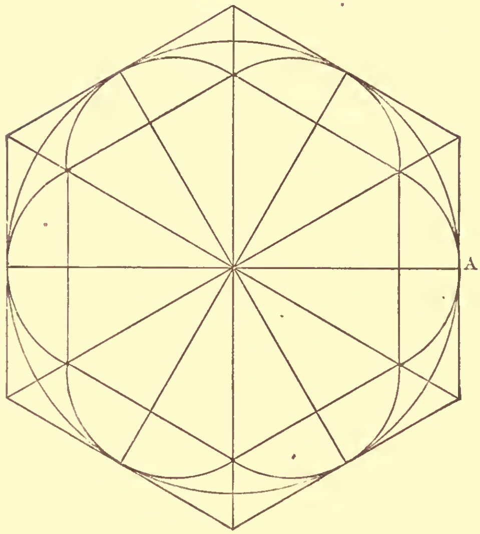
May it not be that they would lead to greater simplicity in the operations and processes to be founded upon them, than the lines now in use?

The two figures, 14 and 15, are very nearly of the same degree of complexity, with the difference apparently in favor of the combination of the circle with squares, but a comparison of the processes by which they may be respectively described on paper will show that the difference is in fact in favor of the combination with hexagons.

Thus, with a pair of compasses set at unity and a parallel ruler, the latter with all the sides of the triangles may be described by means of six points marked by the compasses in the circumference; but to describe the former with the diagonals, either the set of the compasses must be changed after the circle and one diagonal are drawn, or two other points, besides the six points in the circumference, must be located without the circle in order to draw from them the other diagonal, and by means of the diagonals the inscribed and circumscribed squares of the circle.

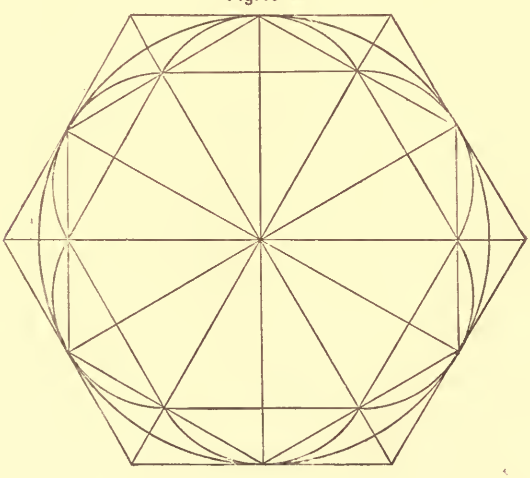
The following figure exhibits the completed circuit of the triangular unit of measure with an arc of its circumscribing circle described upon one of its sides (applicable to the determination of plane angles, as hereinbefore shown), and having a circle and hexagon circumscribed with the sides of the triangles and lines of altitude produced to meet them, as in Figs. 14 and 15.

Fig. 16.



This figure may also be exhibited with additional lines, as follows:

Fig. 16 *a*



The arcs described upon the exterior sides of the regular triangles of which the interior hexagon is composed, are equal in length to the arcs of the circumscribed circle intercepted by the sides of the triangles produced respectively, and the lines of the circumscribed hexagon are tangent to each half of both such arcs. The radius by means of which the arcs are described on the sides of the triangles is one-half the radius by means of which the circumscribed circle is described. The circumscribed hexagon is commensurable with the interior hexagon, the length of the sides of the former being 1,333, that of the latter being 1.

The horizontal diameter of the circle in Fig. 16*a* also (as in the case of Fig. 15*a*) consists of two sides (one each) of two triangles instead of lines of altitude, as in the original phase and in Fig. 14, and the added lines are those directed to be drawn in the description of the second method of applying the triangular unit of measure to the determination of plane angles hereinbefore contained. The hexagon described by all such added lines taken together is equal to the circumscribed hexagon in Fig. 15, and each of such lines is a chord of one-half of the arc of the circumscribing circle of the original regular triangles, and each two of such lines forming one and the same straight line is the chord of an arc of the circumscribed circle in this figure, intercepted by the lines of altitude of two of such triangles produced. Such hexagon is incommensurable except in power with either the interior or circumscribed hexagon.

Might not the study of this figure in both phases lead to valuable results?

But the triangular unit of measure with a complete circumscribed circle instead of one arc thereof as in Fig. 9, page 20, may be applied to the determination of all plane angles as shown in the following figures, in which the angles to be determined are designated by the letters B D′ F, B′ D′ F, &c., B D′ F being the same as B D F in Fig. 9, and to be found as described in the text following that figure.

Fig. 9 a

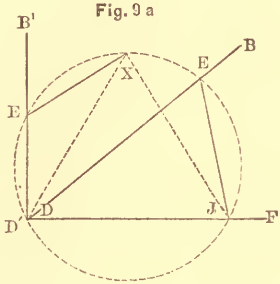


Fig. 9 b

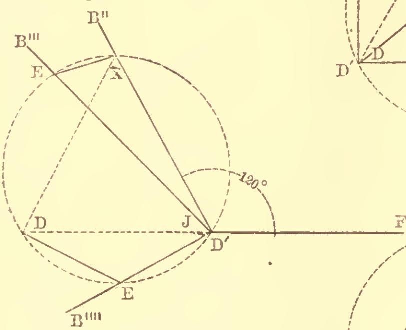
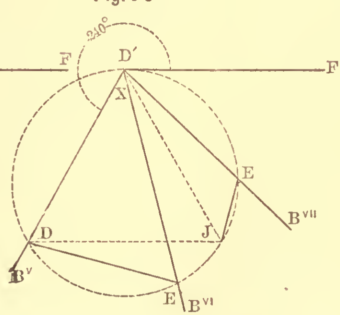


Fig. 9 c

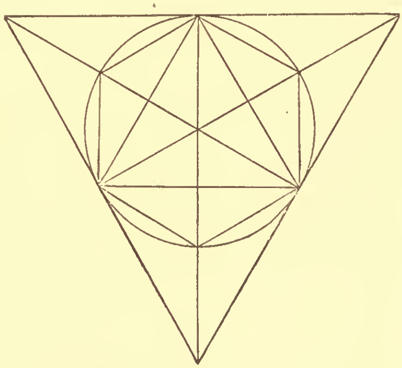


To find B′ D′ F, as in Fig. *9a*, from the point E where the line B′ D′ intersects the arc draw and measure E X. The angle D E X will always be an angle of 120° at whatever point in the arc its vertex may be. Then, by means of the known lines D X and E X and the known angle

D E X, the angle E D X, equal to B′ D′ X, may be determined, and being added to X D J (60°), will be equal to B′ D′ F.

The angles B″ D′ F in Fig. 9*b* and B^{v} D′ F in Fig. 9*c* are measured directly by the unit of measure, and the other angles in those figures may be found in like manner as the angles in Fig. 9*a*.

The following figure exhibits the triangular unit of measure with a circle and regular triangle circumscribed and with lines of altitude produced to the points of the circumscribed triangle, and chords of all the arcs into which the circle is divided by such lines of altitude produced.



The sides of the unit of measure furnish the invariable line instead of the radius of the circle as in Figs. 14 to 16*a*, and the centre of the circle will not lie in the side of any angle measured, greater or less than 30°.

Would not this figure furnish all the requisite trigonometrical functions, and might it not lead to greater simplicity in the operations and processes to be founded upon it than the figures before exhibited ?

§ 10. The oct·astron is the analogue of two independent processes of reasoning, conjoined in the figure, but in nowise connected, each consisting of two sorites combined. In the following figures,

Fig. 17

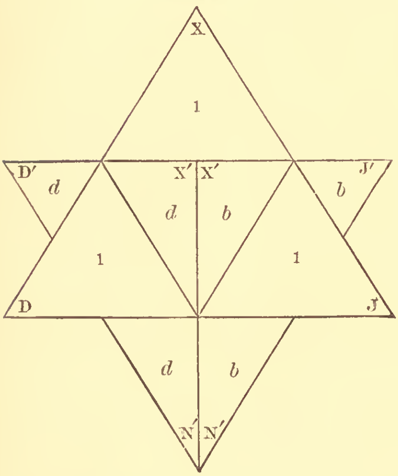
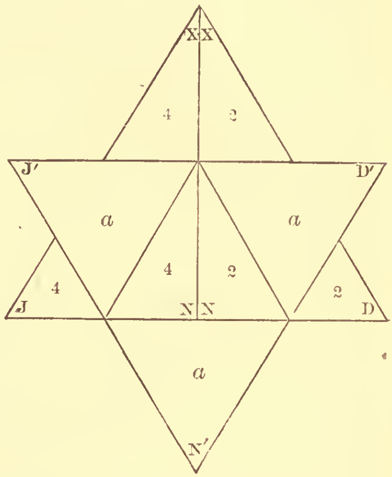


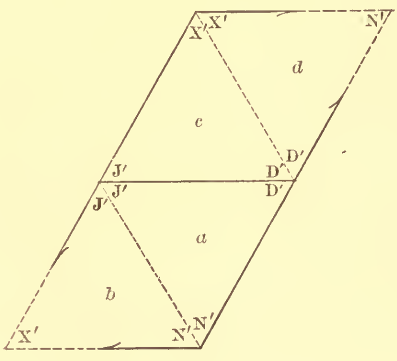
Fig. 18



it is represented in two positions (in each case as seen with the line of vision perpendicular to the vertical axis X N′ at its middle point, the centre of the figure); first, with triangle 1, as in Fig. 3, on page 16, in full view, but of the edge of 2, as the tetrahedron was herein first considered (page 11); and, secondly, after having been turned half-way round on such axis, with triangle *a*, as in the next following figure, in full view.

The following figure represents the second of the intervolved tetrahedra with its faces spread out as a plane in like manner as in Fig. 3, but differs from that, not only in respect to the symbols of the terms, but also in the lateral directions of the two processes of reasoning, descending to the left instead of to the right, and ascending to the right instead of to the left. The tetrahedron formed by the folding of this figure will be the analogue of the sorites considered as beginning in the ascending direction, and the tetrahedron, formed by the folding of Fig. 3, that of the sorites considered as beginning in the descending direction, the letters *a*, *b*, *c*, *d* showing the order of the process in this figure, as the numbers 1, 2, 3, 4 have done in respect to Fig. 3, the triangles *a* and *b* in the former corresponding to 3 and 4 in the latter, and *c* and *d* to 1 and 2.

Fig. 19.



(The words "descending" and "ascending" have been hitherto applied to the processes of reasoning as exhibited on the faces of a single tetrahedron, descending from X to N and ascending from N to X, in both cases referring to a single progressive sorites. Let them be hereinafter considered each as applying to the combined sorites on the faces of each of the tetrahedra intervolved in the oct·astron, viz.: descending throughout, first progressively and then regressively, from X to N, in the first of such tetrahedra, and ascending throughout, in like manner, from N′ to X′ in the second, and let the expression "complete process of reasoning," when hereinafter employed, signify a combination of two sorites descending or ascending throughout, unless it shall be manifest from the context that it is intended to apply only to one.)

By revolving the oct·astron, as held in the position shown in Fig. 17, from left to right, triangles 2 and 4, as in Fig. 3, will successively come in view, and by turning it one-fourth of a revolution with its vertical axis as the diameter of a circle, described by the extremities of such axis with the point X receding from the eye, triangle 3 will come in full view.

By revolving the oct·astron, as held in the position shown in Fig. 18, from left to right, triangles *b* and *d*, as in Fig. 19, will successively come in view, and by turning it with the axis as the diameter of a circle in like manner as before, but with the point N′ instead of X receding, triangle *c* will come in full view.

It will be seen that there are two axes, the extremities of which are designated, one by the symbols X N′ and the other by N X′. No relation between such symbols as they are connected by an axis is demonstrated in either process of reasoning; but that of X an extremity of one axis, with N an extremity of the other, and of N′ an extremity of the former with X′ an extremity of the latter. The reasoning is upon lines wholly on the surface, and not on imaginary lines going through the body of the figure.

When the entire reasoning process in respect to either intervolved tetrahedron shall have been gone through with, there is no going beyond. There is an impassable gulf between the ultimate point of either intervolved tetrahedron and the point of beginning in the other which no process of reasoning unaided by further investigation can span.

§ 11. The oct·astron has hitherto been considered as consisting of two intervolved tetrahedra of the edge of 2, to both of which the included octahedron is common. Let it now be considered as consisting of the octahedron as the primary figure, with tetrahedra of the edge of 1 superposed upon its faces, and let each such tetrahedron be considered as having its points designated on each face by the logical symbols similarly to the two intervolved tetrahedra as hereinbefore shown; that is, four with the symbols X J D N, and considered as being in the descending direction throughout, and four with the symbols N′ D′ J′ X′, and considered as being in the ascending direction throughout; and let such tetrahedra be considered as so superposed that the exterior points of the whole figure shall be designated by the same symbols respectively and relatively to each other, as in Figs. 17 and 18, on page 34. Then will the faces of the whole figure, as brought to view by its revolution, be the same as in those figures and the description following them, except that each face will have three small triangles, with all their points designated, instead of the three considered as one great triangle, with only its exterior points designated, as in the figures.

Let it now be further considered that the designations on the faces of the tetrahedra, which are respectively applied to the octahedron, and also the designations of the vertices of such tetrahedra (points of the oct·astron) opposite such faces are impressed on the faces of the octahedron, on which such faces of the tetrahedra are respectively applied; and let all such tetrahedra be considered as removed.

The faces of the octahedron may then be spread out in several different ways, in each case in two plane figures, of which ways two, with the designations, will be as shown in Figs. 20, 21, 22, and 23, on the next page, to be taken together as they stand, side by side; the designations in the second figure in each case being considered as on the other side of the paper.

If the first of these figures in each case be considered as lifted up to a height equal to the altitude of the octahedron and placed directly over the second, so that the centres of the two middle triangles (3 and *c* in each case) shall be extremities of a line perpendicular to the plane of each such triangle, then the exterior triangles of the upper figure may be folded downward over the points or sides, as the case may be, by which they are connected with the middle triangle, and those of the lower figure upward in like manner, until they respectively meet, and the resulting figure in each case will be the octahedron reconstructed. There are no natural planes passing through the body of the octahedron.

Fig. 20.

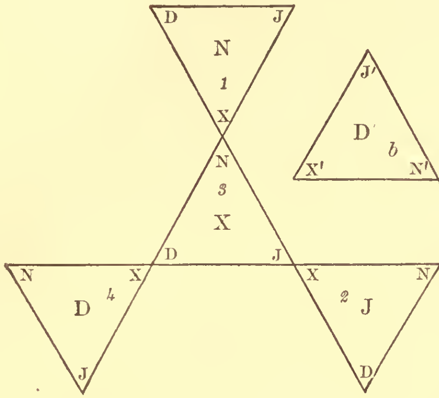


Fig. 21.

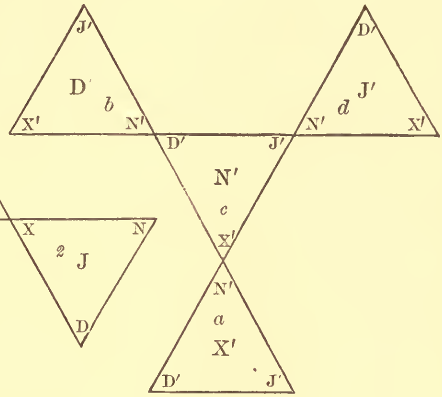


Fig. 22.

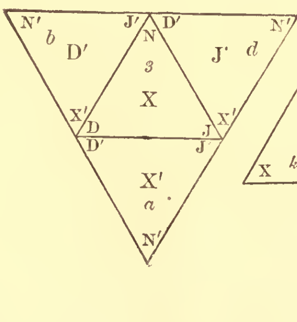
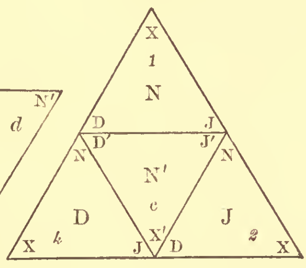
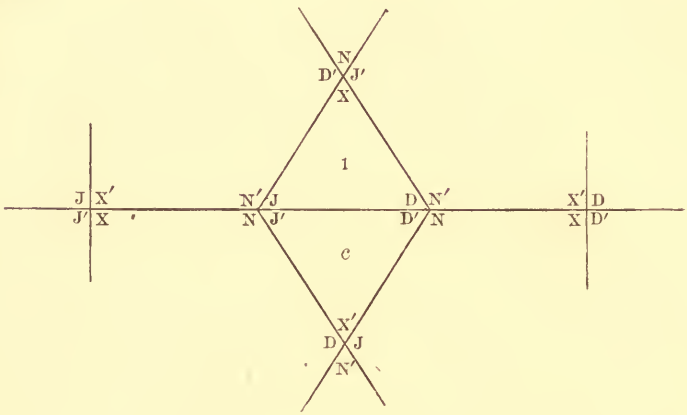


Fig. 23.

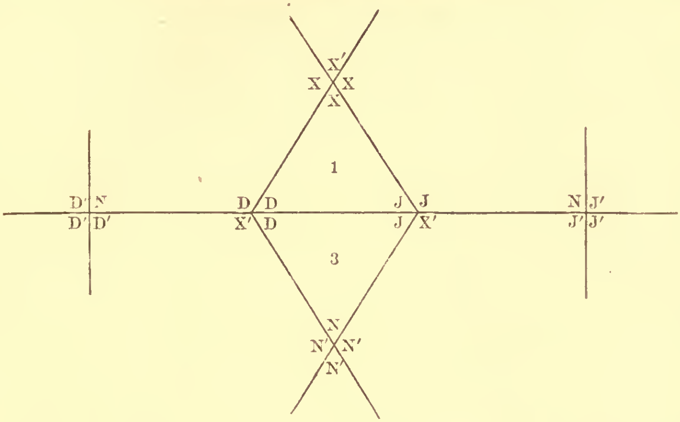


Each point of the octahedron will now be found marked with four different symbols, viz.: two different ones of each process, as shown in the following illustration, in which two faces only of the octahedron appear, the lines produced and otherwise exhibited representing the edges of the octahedron formed by the sides of the adjacent and opposite faces, as will be readily understood.



No two adjacent faces joined by an edge of the octahedron constitute, when spread out, a quadrilateral on which either process is exhibited as a sorites, but in every case a syllogism of one of the processes is conjoined with a syllogism of the other.

If triangle 4 in Fig. 3 (page 16) were turned upward in a semicircle on the point J as a centre, and if triangle *d* in Fig. 19 (page 35) were turned downward in like manner on the point D′ as a centre, the two resulting figures would be in the same forms as Figs. 22 and 23, and could be folded and put together in the form of an octahedron, the points of which would be found marked each with two different symbols, viz.: on one face with a symbol of one process and on each of the other three faces with a symbol of the other process, as follows:



Two sorites only, one of each process, would be exhibited as composed of two syllogisms regularly combined on two adjacent faces, viz.: that on faces 1 and 2 taken together, and that on faces *a* and *b* taken together. The syllogism on each of the other faces would not combine with that on either of the adjacent faces respectively, so as to constitute a sorites.

Figs. 3 and 19 cannot be folded and put together in the form of an octahedron, but either figure and a duplicate thereof may be. The two complete processes of reasoning in such case would be in opposite circular directions.

If the two following figures are folded, the first downward and the second upward (the symbols in the second being considered as on the other side of the paper), and put together so that the edges formed by the meeting of the lines of the openings in the figures shall be opposite each other, the two complete processes, but not combined in regular order as sorites, will occupy four adjacent faces of the octahedron meeting at a common point.

Fig. 24.

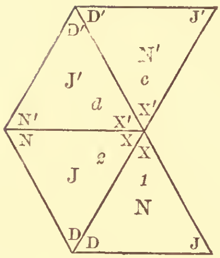
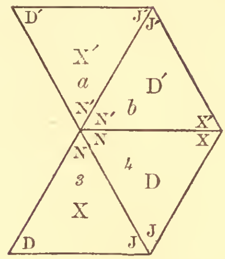
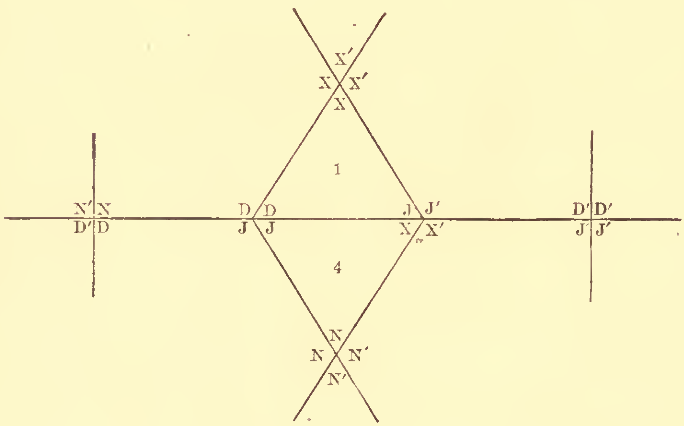


Fig. 25.



The points of the octahedron will now be found marked on their four faces respectively, as follows:



This is, in either case, confusion worse confounded. It is manifest that the octahedron is not the figure designed by nature as the analogue of the perfect and harmonious conjuncture of the two complete reasoning processes, but instead, that the tetrahedron in which it is partially concealed is the analogue of one complete process in respect to four terms (the visible faces of the octahedron in such case having their angles undesignated), and that the octastron in which the octahedron is wholly concealed is the analogue of two complete processes perfectly conjoined, each showing the relation of four terms respectively, but each wholly independent of the other.

If two octahedra with the logical symbols of both the complete processes impressed on their faces, one as secondly and the other as thirdly described, should have the tetrahedra of either one only of such processes superposed upon their appropriate faces, the resulting solid figure in each case, instead of being in the perfect form of a regular tetrahedron of the edge of 2, as it would be as first described, would be irregular, as shown in the following illustrations. The figures are so drawn as to represent the octahedron in each case with one of its axes vertical, the figures being considered as held below the eye. All the symbols except those at the vertices of the superposed tetrahedra and at the centres of those of the visible faces of the octahedron which are in sight in the figures are omitted.

Fig. 26

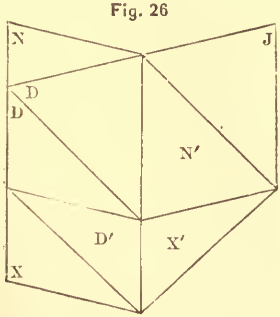
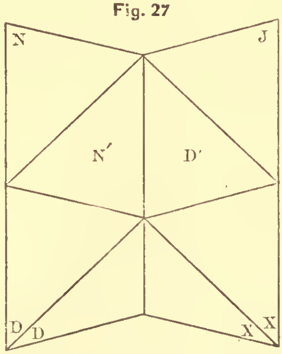


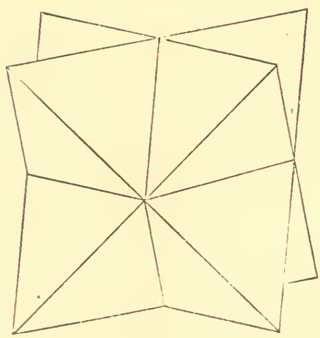
Fig. 27



The significance of the foregoing description of the octahedron (in three phases) showing its inadaptability to be regarded as the analogue of the perfect conjuncture of the two complete processes of reasoning will not appear until the consideration of the sphere is reached. The description has been introduced here as in its appropriate place following the description of the oct·astron showing its perfect adaptability.

§ 12. The oct·astron has hitherto been considered as having one of its axes, X N′, vertical and all the others oblique. Let it now be considered as let fall to one side, in which position all its axes become oblique. It may then appear as shown in the following illustration:

Fig. 28



This figure exhibits the oct·astron in the form not of an outlined but of an out-pointed cube, with intersecting lines connecting diagonally the points of what would be the faces of the cube if it were outlined. The figure may, perhaps, properly be called the skeleton of a cube.

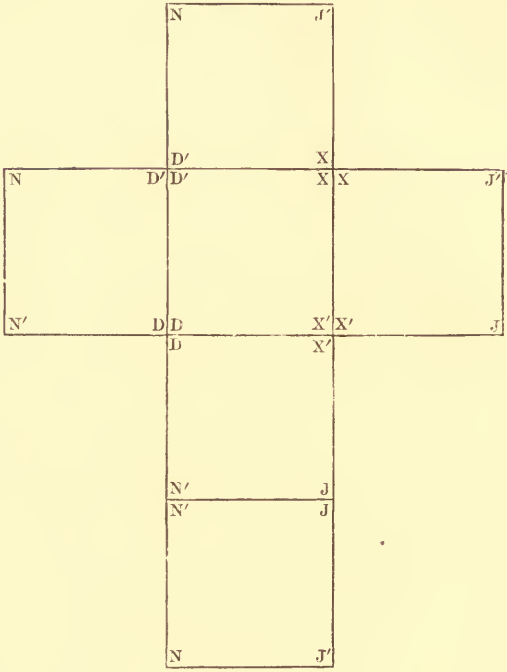
Let the figure now be considered as divided by three planes passing through the centre of the included octahedron in line with its three axes and edges, and there will result eight equal figures, each consisting of a regular tetrahedron, with an irregular right-angled tetrahedron attached to one of its faces.

Twenty-four imaginary right-angled tetrahedra must now be supplied, three to be applied to the faces of the tetrahedron in each of such eight figures, and the figures considered as put together again before a perfect cube of the apparent dimensions of 2, in length, breadth, and height, can be imagined. Such dimensions will not be 2, but $\sqrt{2} = 1.4142$. But the diagonals of the faces will each be 2.

It will thus be seen that, while a regular tetrahedron of the edge of 1 is the nucleus of a cube of the edge of .7071, its edges being diagonals (one of each) of the faces of the cube, as hereinbefore shown, a regular octahedron of like edge (1) is the nucleus of a cube of the edge of 1.4142, its points being the points of intersection of both diagonals of the faces of the cube, and that a regular oct·astron of like edge (1) is also such nucleus, its edges being both diagonals of the faces, and its points the points of the cube; the tetrahedron, the octahedron, and the oct·astron being otherwise than as described wholly hidden within the body of the cube.

If the faces of the cube having its points designated by the logical symbols, as such points are designated on the faces of the included oct·astron as in Figs. 17 and 18, on page 34, be spread out, such designations will be found to be as shown in the following illustration:

Fig. 29

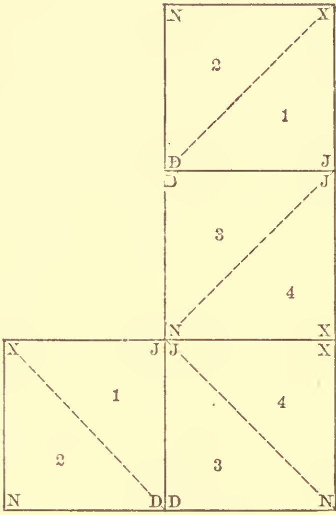


Here again is confusion, even worse confounded than in the case of the octahedron. Two symbols of each process appear on each face, but in no case, considering the terms of the two processes as interchangeable but as retaining their logical significations, can a sorites be found. Two of the faces have each two magnus terms, but no maximus term; two have each two maximus terms, but no magnus term; and the remaining two have each, the magnus and maximus terms, and the two middle terms diagonally opposite respectively, instead of being, the two former, extremes of one of the sides, and the two latter, extremes of the opposite side.

The oct·astron included in the cube as represented in the foregoing figure is considered as consisting of two intervolved tetrahedra, and if diagonals had been drawn on each face of the cube, their point of intersection would have been undesignated. But if the included oct·astron is considered as consisting of tetrahedra superposed on the faces of its included octahedron as described in § 11, on page 37, then it would be necessary to draw such diagonals and their point of intersection on each face would be found designated by eight symbols (different on different faces), two in each of the four triangles into which the face would be divided by such diagonals. Such symbols in the upper face of the cube would be (beginning with the triangle at the left hand and going from left to right) X X′, X′ D, D D′, and D′ X. The confusion in such case would seem to be inextricable. It will be hereinafter shown that the oct·astron must be considered as consisting of tetrahedra both intervolved and superposed.

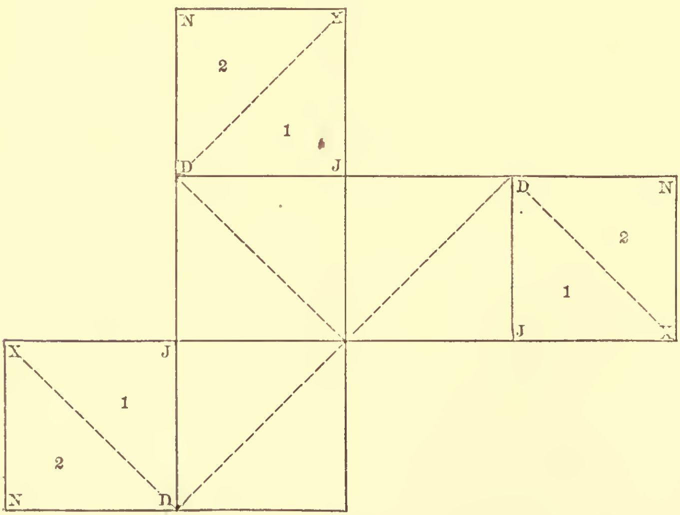
But the faces of the cube may have three combinations of two sorites each, in regular form, as shown in Fig. 30 (next page), in which let the first sorites of each combination be considered as in the descending direction and the second in the ascending (instead of being descending or ascending throughout), each combination beginning at a different point from either of the others, but all terminating at the same point. If a regular tetrahedron be considered as the nucleus of the cube, then assuming the diagonals of the three faces on which are given the three sorites in the descending direction to be three edges of such tetrahedron, the diagonals of the other three faces are not the other three edges.

Fig. 30



The following figure

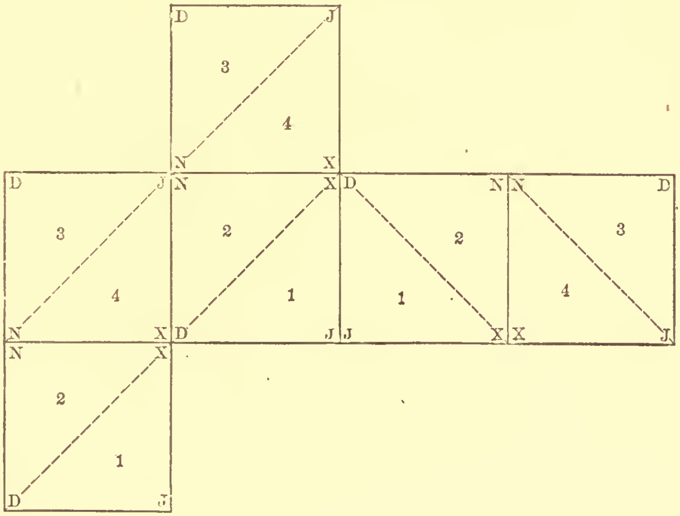
Fig. 31



is the same as the preceding in respect to such first-mentioned three faces, but in respect to the other three the diagonals are in the opposite directions respectively, and are edges of the included tetrahedron, but the sorites in the ascending direction cannot be represented upon such faces respectively, except by transposing the terms as they appear in Fig. 30, in which case the two sorites of each combination would be found to run counter to each other, in opposite circular directions.

In neither of the combinations shown in Fig. 30 does the second sorites return to the point of beginning of the first, as in the case of the combination on the faces of the tetrahedron. But they may be exhibited in such manner, as in the following illustration,

Fig. 32



that in each case the second sorites shall return to the point of beginning of the first, but such point will not be the same in respect to any two combinations.

In the foregoing figure it will be seen that each of the combinations has the line X N or N X (analogue of the ultimate conclusion of each sorites) common, instead of the line J D or D J (analogue of the middle premise in each sorites), which was the only common line capable of actual measurement in the original construction and combination of the figures.

In like manner as before, three only of the diagonals are lines of edges of the included regular tetrahedron.

To return to the oct·astron. In one or the other of the two aspects in which it has been exhibited—that is, either first as consisting of a combination of two intervolved tetrahedra or of eight tetrahedra superposed upon the faces of an octrahedron, or secondly as the nucleus of a cube (its dissection as described in § 4 being taken in connection with the second)—must the oct·astron be regarded in order to find a unit upon which to base the operations and processes of geometry looking to the measurement of its volume. In the first, the analysis is along natural lines and actually existing planes lying wholly on the surface (each tetrahedron, either intervolved or superposed, being considered by itself); in the second, with the exception of the diagonals, it is wholly along imaginary lines and planes, some of which lie wholly on the imagined surface, and the others wholly within the body of the figure.

Which of the two aspects is the simpler and the more in accord with nature?

§ 13. Twelve octahedra (edge 1) may be superposed upon the twenty-four faces of the oct·astron, each octahedron superposed upon or perhaps rather interposed between two adjacent faces, one each of two of the tetrahedra (edge 1) of the oct·astron, and twenty-four tetrahedra may be interposed, each between two adjacent faces, one each of two of such octahedra, one edge of each of such interposed tetrahedra falling upon and coinciding with an edge of one of the tetrahedra of the oct·astron, and six octahedra may be interposed, each between four faces, one each of four of such interposed tetrahedra meeting at a common point, and the resulting figure will be a regular octahedron of the edge of 3. One point of the included and wholly hidden oct·astron will be at the centre of each face of such octahedron.

It is now manifest that such octahedron will be the nucleus of a second oct·astron of the edge of 3, composed of two intervolved tetrahedra of the edge of 6, and that such second oct·astron will be the nucleus of a third octahedron of the edge of 9, which will be the nucleus of a third oct·astron of the edge of 9, composed of two intervolved tetrahedra of the edge of 18, and so on, *ad infinitum.*

The first octahedron (the central figure of the first oct·astron) may be called an octahedron of the first order, and the tetrahedron and oct·astron formed thereon a tetrahedron and an oct·astron of the first order, and the second of each, of the second order, and so on.

The primary figure is evidently the octahedron on which both the others are constructed.

The following table exhibits the edge and volume of each figure up to and including the sixth order:

TABLE OF EDGES AND VOLUMES

OF THE OCTAHEDRON, TETRAHEDRON, AND OCT·ASTRON, BEGINNING WITH UNITY AS THE EDGE OF THE OCTAHEDRON.

ORDER.		OCTAHEDRON.	TETRAHEDRON.	OCT·ASTRON.
1	Edge	1	2	1
	Vol.	4	8	12
2	Edge	3	6	3
	Vol.	108	216	324
3	Edge	9	18	9
	Vol.	2916	5832	8748
4	Edge	27	54	27
	Vol.	78732	157464	236196
5	Edge	81	162	81
	Vol.	2125764	4251528	6377292
6	Edge	243	486	243
	Vol.	57395628	114791256	172186884

The edge of the tetrahedron of each order is in all cases double that of the octahedron and oct·astron of the same order.

The volume of the tetrahedron of each order is double and that of the oct·astron three times that of the octahedron of the same order.

The edge of each figure of each order after the first is 3 times and the volume 27 times that of the same figure of the next preceding order.

The volume of the octahedron of each order after the first is $13\frac{1}{2}$ times that of the tetrahedron and 9 times that of the oct·astron of the next preceding order; and the volume of the tetrahedron of each order after the first is 54 times and that of the oct·astron 81 times that of the octahedron of the next preceding order.

To find the edge and volume of the octahedron of any order, take the number of the next preceding order as the exponent of a power, and raise 3 and 27 to such power. The power of 3 will be the edge, and that of 27 multiplied by 4 will be the volume required.

Thus, the edge and volume of each figure of the tenth order are as follows:

ORDER.		OCTAHEDRON.	TETRAHEDRON.	OCT·ASTRON.
10	Edge	3^9	$2(3^9)$	3^9
	Vol.	$4(27^9)$	$8(27^9)$	$12(27^9)$

The volume of each figure of any order being 27 times that of the same figure of the next preceding order, the difference between the volumes in any two consecutive orders will, of course, be the product of the volume in the first of such orders multiplied by 26. Such difference, except in the case of the oct·astron, is also equal to the product of 26 multiplied by twice the difference between the numbers of the edges of the same figure of the two orders, and again by 9 raised to a power, the exponent of which is equal to the number of the first of such orders — 1.

Thus, such differences in respect to the octahedron and tetrahedron up to the sixth order are as follows:

Nos. of Orders.	Differences between Volumes.	
	Octahedra.	Tetrahedra.
1st *and* 2d	$26 \times 4 \times 9^0$	$26 \times 8 \times 9^0$
2d " 3d	$26 \times 12 \times 9^1$	$26 \times 24 \times 9^1$
3d " 4th	$26 \times 36 \times 9^2$	$26 \times 72 \times 9^2$
4th " 5th	$26 \times 108 \times 9^3$	$26 \times 216 \times 9^3$
5th " 6th	$26 \times 324 \times 9^4$	$26 \times 648 \times 9^4$

and so on.

To apply the above process to the oct·astron, it is necessary to multiply the difference between the numbers of the edges given in the table also by 3.

The author confesses himself to have been and still to be in a quandary as to whether the oct·astron of the first order should be described as of the edge of 2, being that of each intervolved tetrahedron, or of 1 being that of each superposed tetrahedron. As some description seemed to be necessary, the latter has been adopted. It is manifest that it cannot be regarded as of the edge of 3.

The tetrahedron of the first order in the table is of the edge of 2. Let it now be considered that it is in fact a tetrahedral yard, each edge being one yard in length. The edge of each smaller tetrahedron, of four of which it in part consists, will then be half a yard or one and a half feet, and let it be considered that it is desired to ascertain the contents of each of the three figures of the first order, and then of the second and succeeding orders in the table in terms of tetrahedral feet.

The edges of the figures of the first order in such terms will be 1.5, 3, and 1.5 respectively. To ascertain the volumes find the third power of $1.5 = 3.375$, and multiply such third power by the volumes in the first order, as in the table, viz.: 4, 8, 12. To find the edges and volumes in the second order multiply the edges in the first order as above by 3, and the volumes as so found by 27, and proceed in like manner to find the edges and volumes in the third order, and so on.

Similarly the volumes could be found with 2, 2.5, or any number, whole or fractional, as the edge of the octahedron of the first order.

If a table were constructed upon any such octahedron of greater or less edge than 1, as of the first order, the edges and volumes of the figures of all the orders would have the same definite relations to the edges and volumes of the figures of the corresponding orders in the foregoing table throughout, as in the first order. No such table would, therefore, be required.

In all cases where the edge of the tetrahedron is an odd number, procedure in physical construction upon the octahedron of edge 1 as the central figure would be upon artificial lines and faces produced by forced sections of the regular figures. While this is practicable by means of fractions in arithmetic, which has to do only with abstract numbers, it would be utterly impracticable in physical geometry.

In what way the building up of the figures and the table may serve in geometrical processes the author is unable to say, but he will hereinafter show that the tetrahedra in an octahedron of the second order are analogues of compound logical processes through which the two complete simple processes on the faces of the oct·astron of the first order are brought into perfect union. The table will be herein called the table of natural involution.

If a regular tetrahedron (edge 1) be taken as the central figure, and be built upon by superposing octahedra upon its faces, and interposing tetrahedra between the faces of such octahedra, the resulting figure will be found to be an irregular octahedron having four of its faces regular hexagons of side 1 (a point of the central figure being at the centre of each of such faces), and the other four, regular triangles of side 1. By superposing regular tetrahedra upon the latter four, the further resulting figure will be a regular tetrahedron of edge 3.

This figure could be again in like manner built upon, and the second ultimately resulting figure would be a regular tetrahedron of edge 9, and so on. But the first resulting figure would in all cases be irregular.

If a table were constructed upon such a series it would be one of tetrahedra only, the edge of which in each order would be one-half and the volume one-eighth those of tetrahedra of the corresponding orders in the table of natural involution. Such table would therefore not be required.

§ 14. A single regular tetrahedron, the unit of measure, may be divided by four sections into five fractional parts, of which four will be each the one-eighth part, each in the form of a regular tetrahedron of edge $\frac{1}{2}$ and volume $\frac{1}{8}$, and the fifth, the one-half part, a regular octahedron of the same edge, and equal in volume to the other four parts combined.

This octahedron may now be subdivided by dissection into fifty-one parts (equal to the number of figures in an octahedron of the second order), of which, nineteen will be regular octahedra (one of which will be the central figure), and thirty-two regular tetrahedra. The edge of each such tetrahedron will be $\frac{1}{6}$, and the volume $\frac{1}{216}$ of the edge and volume of the original unit.

In like manner the central octahedron may be subdivided, and the edge of each smaller tetrahedron will be $\frac{1}{18}$, and the volume $\frac{1}{5832}$ of the edge and volume of the original unit, and so on.

If the table of natural involution be considered as extended in the opposite direction from the point of beginning, in respect to the tetrahedron only, beginning with a regular tetrahedron of edge 1 divided as above described, and the orders correspondingly numbered backward (or more properly in the descending direction), the edge and volume of the tetrahedron of each order will be reciprocals respectively of the edge and volume of the tetrahedron of the order of corresponding number in the forward (ascending) direction.

The centre, it is obvious, can never be reached in the descending direction, how far soever the process may be continued. The central figure will always be that of an octahedron. If it be attempted to divide the octahedron in any other manner so as to reach the centre, the resulting figures will be irregular, and irregularity in figure must, as it would seem to the author, necessarily involve intricacy in calculation.

Does not this, in connection with what has been hereinbefore shown as to the perfect accord between the logical and geometrical processes along the lines of the faces of the tetrahedron, seem to make manifest that nature forbids any attempt to reach the centre of a solid figure, and that all processes relating thereto should be conducted along natural lines and planes lying on surfaces as originally existing or as superposed, or by natural sections disclosed, and not along lines and planes produced by forced sections seeking to reach the centre?

If a regular tetrahedron of edge 1, the unit of measure, be considered as divided in the reverse direction of the process of building upon it as the central figure, as before described, such division would necessarily begin with cutting off from each point a regular tetrahedron of edge $\frac{1}{3}$ and volume $\frac{1}{27}$, leaving an irregular octahedron for further division, as described in the building-up process. The tetrahedron to be cut off in the second instance would be of edge $\frac{1}{9}$ and volume $\frac{1}{729}$, and so on.

If the table suggested in connection with the description of such building-up process (but which would never be required) be considered as extended in the opposite direction, in like manner as described in respect to the extension of the table of natural involution, the edge and volume of the unit would constitute the first order in each direction, and the edge and volume in any order in either direction would, in like manner, be reciprocals respectively of the edge and volume in the order of corresponding number in the other direction.

This, in respect to the edge and volume of the unit constituting the first order in the descending direction, may at first seem paradoxical, but a comparison of the two tables will show that it is true, and further, that it is requisite in order that the relations of the two tables to each other in both directions, and of each order in either table in one direction to the corresponding order in the same table in the other direction, may be symmetrical throughout.

In the description of the suggested table (based upon the tetrahedron as the central figure) in the ascending direction, it was stated that the edge of the tetrahedron of each order therein would be one-half and the volume one-eighth that of the tetrahedron of the corresponding order in the table of natural involution.

This is exactly reversed in the extension of the two tables in the descending direction. The edge of the tetrahedron of every order, beginning with the first in the descending direction of the suggested table is twice and the volume eight times the edge and volume respectively of the tetrahedron of the corresponding order in the descending direction of the table of natural involution.

The two tables in both directions up to and including the fourth order, are as follows, the table of natural involution being exhibited only as to the tetrahedron:

Order.		Table of Natural Involution. Of the Tetrahedron only.		2d Table as Suggested. Of the Tetrahedron.	
		Edge.	*Volume.*	*Edge.*	*Volume.*
Ascending.	4	54	157464	27	19683
	3	18	5832	9	729
	2	6	216	3	27
	1	2	8	1	1
Descending.	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{1}$	$\frac{1}{1}$
	2	$\frac{1}{6}$	$\frac{1}{216}$	$\frac{1}{3}$	$\frac{1}{27}$
	3	$\frac{1}{18}$	$\frac{1}{5832}$	$\frac{1}{9}$	$\frac{1}{729}$
	4	$\frac{1}{54}$	$\frac{1}{157464}$	$\frac{1}{27}$	$\frac{1}{19683}$

The tetrahedron is the principal figure in the table of natural involution, the octahedron being the primary figure upon which it is constructed, and the oct·astron, as it were, a compound tetrahedron, wholly lost sight of in the descending direction of the table.

The unit 1 has no place in the table, considered with reference to the tetrahedron only, but is the base of the first order in both directions, in which the edges are both its product and quotient by 2, and in the full table it has place only as the representative of a line which is no real but simply an ideal thing.

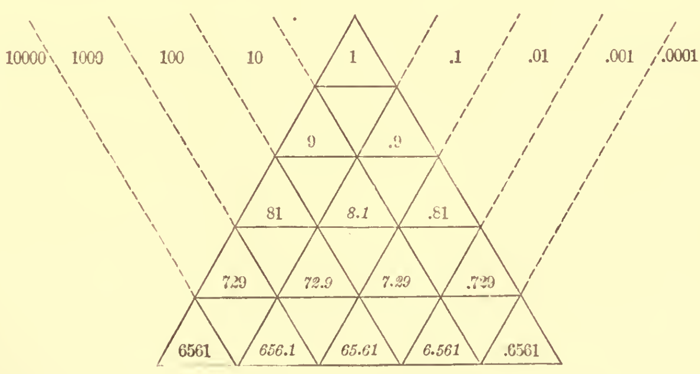
If the orders of the decimal scale had been in like manner numbered, with the unit 1 regarded as of the first order in both directions, each subsequent order of corresponding number in both directions would have been reciprocals of each other. Thus,

DECIMAL SCALE

(As suggested).

Ascending.						Descending.				
6th Order.	5th Order.	4th Order.	3d Order.	2d Order.	1st Order in both Directions.	2d Order.	3d Order.	4th Order.	5th Order.	6th Order.
100000	10000	1000	100	10	1	.1	.01	.001	.0001	.00001

That the first order should occupy but one place as the base in both directions instead of two, as in the tables of tetrahedra above given, is, as it would seem to the author, fairly to be inferred upon consideration of the following diagram in the form of a regular triangle in which, with the unit at the vertex, the first terms of the successive orders of difference in both directions are shown to be connected laterally by a regular gradation of intermediate differences.



Is not the difference between the tables and the scale in this respect to be accounted for by the fact, that in the two former the thing considered as the base in each direction is concrete (whether regarded either as an actual solid or simply as a volume of space), but in the latter is abstract?

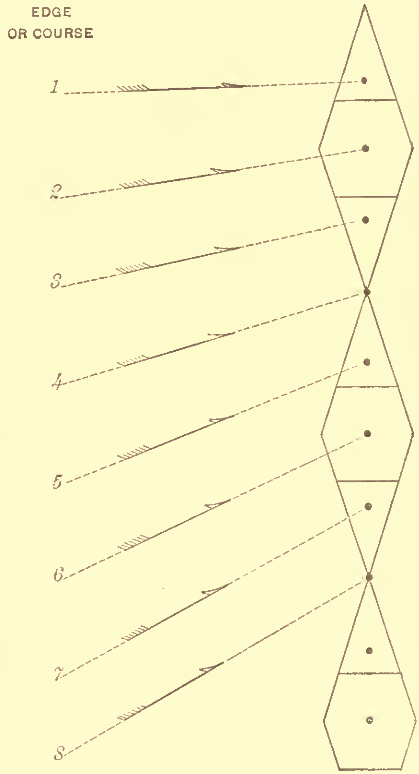
From the foregoing considerations it would seem that the unit 1 is simply the base of all numbers and quantities, by means of which they are measured, and cannot itself be regarded as in any sense a number.

The central figure of any course of a regular tetrahedron, the number of which is 3 or $3 + 1$, or any multiple of 3, or any such multiple $+ 1$, is a tetrahedron, and that of any course the number of which is $3 - 1$, or any multiple of 3, $- 1$, is an octahedron.

The central figure of a right triangular pyramid the number of courses of which is any odd number, is a regular tetrahedron, and that of one the number of courses of which is 2 or the sum of 2 and 4, or of 2 and any multiple of 4, is a regular octahedron, and the centre of such figure in either case is the centre of the pyramid. But if the number of courses is 4 or any multiple of 4, there is no central figure, and the centre of the pyramid in such case is at the point of intersection of four planes passing through the pyramid parallel to and equidistant from its four faces respectively, and each between two courses relatively to one of the faces and its opposite point. Such point of intersection is common to fourteen figures (edge 1) viz.: six octahedra and eight tetrahedra, which together constitute a regular octahedron of edge 2 and volume 32. Such edge is twice and volume eight times the edge and volume of the octahedron of the first order in the table of natural involution.

Thus it appears that in such case nature does not forbid an attempt to reach the centre. But she permits it only in the ascending direction, and in cases only in which it can be accomplished by a process regular throughout.

The foregoing observations will more clearly appear by the accompanying diagram, showing a section of the central figures in the several courses made by a plane passed perpendicularly from the vertex to the base of a pyramid of edge 8, and considered also as a series of pyramids of edges from 1 up to 8, the arrows pointing to the centre of a pyramid of each edge consecutively. The centre of the pyramid is at a point in its line of altitude three-fourths the length thereof from the vertex and one-fourth from the base.



The number of the course in which, or of the first of the two courses in the plane dividing which, the centre of a pyramid of any given edge is to be found may be ascertained by subtracting from the number of the given edge the whole number contained in the quotient arising from the division of the number of the given edge by 4. The remainder will be the number of the course required.

Thus the centre of each, successively, of a series of pyramids of edges as follows will be

Of edge 159 in a tetrahedron in course 120,
Of edge 160 between courses 120 and 121,
Of edge 161 in a tetrahedron in course 121,
Of edge 162 in an octahedron in course 122,
Of edge 163 in a tetrahedron in course 123,
Of edge 164 between courses 123 and 124,
Of edge 165 in a tetrahedron in course 124,

and so on.

§ 15. The powers of numbers were undoubtedly first derived from the consideration of a square and cube divided into smaller equal squares and cubes, of which one in each case was regarded as the unit of measure, and hence the name of square for the second power and of cube for the third. All higher powers are mere multiples.

Such powers, considered as involving the notion of space, are the same and calculable in like manner in the case of the regular triangle divided into smaller equal regular triangles, as in the case of the square; and in the case of the regular tetrahedron, considered as divided into smaller equal volumes of regular tetrahedra, as in the case of the cube.

This will be evident as to the regular triangle upon mere inspection of such a triangle so divided. But it will not be so in the case of the regular tetrahedron.

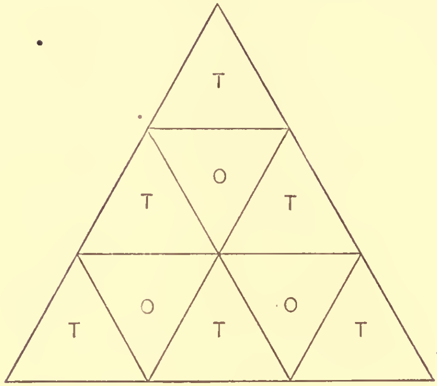
Let a regular tetrahedron of the edge of 1, the unit of measure, be regarded as standing, not upon either face as a base, but upon one of its points with the opposite upturned face horizontal.

This will be the first course of a regular tetrahedron or inverted right triangular pyramid of any number of uniformly increasing courses to be superposed thereon successively.

In constructing the second course, the first figure will obviously be an octahedron to be superposed on the upturned face of the single tetrahedron of the first course. On the three lateral faces of the octahedron pointing downward let there be superposed three tetrahedra and the second course will be complete. Represented by numbers, it consists of the octahedron equal to 4 tetrahedra, and the 3 superposed tetrahedra, making the volume of the second course equal to 7 tetrahedra, which, added to the number in the first course, 1, makes $8 = 2 \times 2 \times 2$.

Thus far all seems simple enough, but when the next course is considered there will be found a unit which has no visible representation on the external face. The following figure exhibits such external face, each smaller triangle being marked T or O, to signify that it is the face of T, a tetrahedron, or O, an octahedron.

Fig. 33.



Here are three octahedra and six tetrahedra. The volume of the course, as calculated from an external view, would be $3 \times 4 + 6 = 18$, to which add the volume of the two preceding courses, 8, and the total apparent volume of the pyramid of three courses is 26.

But by examining the other side of the third course, the lateral faces (one of each) of the three octahedra meeting at their common point in the centre of the external face, as above shown, will be found diverging from such point along the three coinciding lateral edges of such octahedra until they meet and their lower horizontal sides coincide with the sides of the upper horizontal face of the single octahedron in the preceding course, making a volume of space equal to the unit of measure. This unit added to the volume found as above, makes the total actual volume of the tetrahedron whose edge is 3, $27 = 3 \times 3 \times 3$.

In the fourth course the number of such concealed units will be 3, in the fifth 6, in the sixth 10, and so on, in a series, the differences of the terms of which increase by 1 as each course is successively superposed.

The volumes of the several courses constitute a series, the first order of differences of which begins with 6 and increases in multiples thereof by 2, 3, 4, and so on, the second order of differences being, of course, in each case 6. Thus,

Series.	1,		7,		19,		37,		61,		91, &c.	
1st Order of Differences.		6,		12,		18,		24,		30, &c.		
2d Order of Differences.			6,		6,		6,		6, &c.			

Such series cannot be found in the courses of a cube all of which, considered with reference to any one side as a base, are equal to each other, except by considering and taking together for each term of the series consecutively the whole course on one side of the cube, a part of the corresponding course on a second side, and a still less part of the corresponding course on a third side, the three sides having a common point. But such series, omitting the first term, is the first order of differences of a series of volumes of cubes beginning with 1.

§ 16. The following table exhibits an analysis of the several courses of a regular tetrahedron of the edge of 12, and also of a series of regular tetrahedra (whole figures) of edges from 1 up to 12.

ANALYSIS OF REGULAR TETRAHEDRON.

No. of Course and Edge of Figure.	Of Courses.				Of whole Figures.			
	Tetrahedra.	Octahedra.	Vol. in Tet.	Total Vol.	Tetrahedra.	Octahedra.	Vol. in Tet.	Total Vol.
1	1 +	(0 × 4 =	0) =	1	1 +	(0 × 4 =	0) =	1
2	3 +	(1 × 4 =	4) =	7	4 +	(1 × 4 =	4) =	8
3	7 +	(3 × 4 =	12) =	19	11 +	(4 × 4 =	16) =	27
4	13 +	(6 × 4 =	24) =	37	24 +	(10 × 4 =	40) =	64
5	21 +	(10 × 4 =	40) =	61	45 +	(20 × 4 =	80) =	125
6	31 +	(15 × 4 =	60) =	91	76 +	(35 × 4 =	140) =	216
7	43 +	(21 × 4 =	84) =	127	119 +	(56 × 4 =	224) =	343
8	57 +	(28 × 4 =	112) =	169	176 +	(84 × 4 =	336) =	512
9	73 +	(36 × 4 =	144) =	217	249 +	(120 × 4 =	480) =	729
10	91 +	(45 × 4 =	180) =	271	340 +	(165 × 4 =	660) =	1000
11	111 +	(55 × 4 =	220) =	331	451 +	(220 × 4 =	880) =	1331
12	133 +	(66 × 4 =	264) =	397	584 +	(286 × 4 =	1144) =	1728

Let n signify the number of any course or the edge of any whole figure, s the solidity of any course, and S the solidity of any whole figure.

The number of tetrahedra in any course is equal to $n(n-1)+1$, and also to $\frac{(n-1)^2+(n+1)^2}{2}-n$.

The number of octahedra in any course is equal to $\frac{n(n-1)}{2}$, and is also equal to one-half the number less 1 of tetrahedra in the same course.

The volume of the octahedra in any course is equal to double the number of tetrahedra in the same course, less 2.

The volume of any course is equal to 3 times the number of tetrahedra in the same course, $-$ 2. It is also equal to 6 times the number of octahedra in the same course, $+$ 1. The expression in either case may be reduced to

$$s = 3n(n-1) + 1.$$

It is also equal to twice the sum of the numbers of both figures in the same course, $-$ 1.

To find the numbers of the tetrahedra and octahedra of which a regular tetrahedron of any given edge (n) or of any given volume (S) consists, it is necessary to begin with the octahedra.

The number of octahedra in a regular tetrahedron of any given edge is equal to $n^2 \times \frac{n}{6} - \frac{n}{6}$, and of any given volume is equal to $\frac{S}{6} - \frac{n}{6}$, but n in the second expression is an unknown quantity.

The number of tetrahedra in a regular tetrahedron of any given edge is equal to $2\left(n^2 \times \frac{n}{6} - \frac{n}{6}\right) + n$, and of any given volume is equal to $2\left(\frac{S}{6} - \frac{n}{6}\right) + n$, but n in the second expression is an unknown quantity.

The volume of any regular tetrahedron is equal to

3 times the number of the tetrahedra therein contained, $-2n$, and also to 6 times the number of the octahedra therein contained, $+n$.

It may be expressed in either case thus:

$$S = 6\left(n^2 \times \frac{n}{6} - \frac{n}{6}\right) + n.$$

It is also equal to twice the sum of the numbers of both figures therein contained, $-n$.

It is also equal to the product of the number of tetrahedra in the n^{th} course multiplied by $(n+1), -1$.

It is also equal to the product of the number of octahedra in the n^{th} course multiplied by $4\left(\frac{n}{2} + \frac{1}{2}\right) + n$.

The value of $\frac{n}{6}$ is equal to $\sqrt[3]{\frac{S}{6^3 = 216}}$.

The value of n, when the volume of a regular tetrahedron only is given, may be found by the usual arithmetical process of extracting the third root, or by a very much simpler [illegible]d, with the aid of the following table, in which [illegible] as filled up and made complete) is shown the nu[illegible]tahedra in every regular tetrahedron, the quotien[illegible]e of which, divided by 6, is a whole number from [illegible]7.

SEE POSTSCRIPT.

Let E signify the num[illegible]dge or root in the last column, such number as e[illegible] the process to be shown being approximate, n bein[illegible]bol of the required edge or root when found.

The numbers shown in the first column of differences are negative quantities, and those in the second column, positive.

TABLE

OF OCTAHEDRA (EDGE 1) CONTAINED IN REGULAR TETRAHEDRA OF EDGE GIVEN IN LAST COLUMN.

Value of $\frac{n}{6}$ or $\frac{E}{6}$.	No. of Octahedra.	Differences between Nos. Oct. in Regular Tetrahedra of Edges. E and E − 1.	E and E + 1.	Edge or Root. n when known. E approxim'te
1	35	— 15	21	6
2	286	— 66	78	12
3	969	— 153	171	18
4	2300	— 276	300	24
5	4495	— 435	465	30
6	7770	— 630	666	36
7	12341	— 861	903	42
8	18424	— 1128	1176	48
9	26235	— 1431	1485	54
10	35990	— 1770	1830	60
11	47905	— 2145	2211	66
12	62196	— 2556	2628	72
*	* *	* *	* *	*
99	34930665	— 176121	176715	594
100	35999900	— 179700	180300	600
101	37090735	— 183315	183921	606
*	* * *	* *	* *	*
274	740549390	— 1350546	1352190	1644
*	* * *	* *	* *	*
793	17952380459	— 11316903	11321661	4758
*	* * *	* *	* *	*
1216	64729643840	— 26612160	26619456	7296
*	* * *	* *	* *	*
1667	166766685001	— 50015001	50025003	10002

To find, by means of the foregoing table, the edge of any regular tetrahedron of which the volume only is given, or to find the third root of the greatest third power and remainder over, if any, contained in any given number, divide the given number by 6, find in the table the number of octahedra nearest the quotient and subtract such number from the quotient. The remainder, when the nearest number exceeds the quotient, will be a negative quantity. If there are two numbers equally near the quotient, either may be taken. This can never occur when the given number is an exact third power.

Note the number of the edge in the same line with the nearest number taken.

Observe now whether the remainder exceeds, if it be a positive quantity, the number shown in the second column of differences in the table, or, if it be a negative quantity, that shown in the first column, and if it does, then subtract therefrom, if it be positive, the difference between the numbers of octahedra in E and E + 1, being the number shown in the second column of differences in the table; but if it be negative, then the difference between the numbers of octahedra in E and E − 1, being the number shown in the first column of differences, and to be considered as a negative quantity, and continue subtraction successively, if necessary, namely, in the first case, of differences between the numbers of octahedra in E + 1 and E + 2, and between those in E + 2 and E + 3 (not shown in the table but readily found), or in the second case, of differences between the numbers of octahedra in E − 1 and E − 2 and between those in E − 2 and E − 3 (also not shown in the table but readily found), until the remainder in the first case shall be less than the next difference, or, in the second case, shall become a positive quantity, equal to or greater than the quotient of the number of the least edge which shall have come into the process divided by 6. Then, from such remainder, subtract the quotient by 6 of the number of the greatest edge which shall have come into the process (that is, $\left(\frac{E + 3 \text{ or } 2 \text{ or } 1}{6}\right)$ in the first case, or of the least edge $\left(\text{that is, } \frac{E - 3 \text{ or } 2 \text{ or } 1}{6}\right)$ in the second case.

The number of the greatest edge in the first case or least in the second, viz., $E + 3$ or 2 or 1, or $E - 3$ or 2 or 1, as the case may be, will be the required root. If there be no remainder, the given number will be a perfect third power; but if there be, then multiply the remainder by 6, and the product will be the excess of the given number over and above the greatest third power therein contained, and will be the last remainder that would be found in the usual arithmetical process of extracting the third root.

If the first remainder, namely, that arising from the subtraction of the nearest number from the quotient, does not exceed the difference in either case, as hereinbefore directed to be observed, then E will be the required root, provided such remainder be equal to or exceed $\frac{E}{6}$, and the given number will be a perfect third power if it be equal; but if it exceed, then subtract therefrom $\frac{E}{6}$ and multiply the remainder by 6, and the product will be the excess of the given number over the greatest third power therein contained.

If the first remainder be less than $\frac{E}{6}$, then E − 1 will be the required root, and the process is to be continued in like manner as before described in the case where the first remainder is a negative quantity, although such remainder in this case will not be a negative quantity unless the given number be less than $E^3 - E$. The point at which the first remainder changes from one kind of quantity, positive or negative, to the other, or rather becomes 0, is that where the given number is $E^3 - E$. The point at which subtraction from the first remainder begins with a positive quantity, and is continued thereafter with positive quantities on the one side, and on the other begins with a negative quantity, and is continued thereafter on the same side with negative quantities, is not that at which the first remainder changes from one kind of quantity, positive or negative, to the other, but that where the given number is E^3.

To illustrate. Let the given volume of a regular tetrahedron of which the edge is required be . . . 729

Divide it by 6. Quotient	121.5
Find and subtract nearest number of octahedra in table, noting edge (E . 6) . . .	35
First remainder	86.5
Subtract difference between numbers of octahedra in E (6) and E + 1 (7), as shown in table	21
Remainder	65.5

Subtract diff. bet. nos. oct. in E + 1 (7) and E + 2 (8), to be found as follows:		
Difference taken from table as above	21	
+ E + 1 = ...	7	28
Remainder		37.5
Subtract diff. bet. nos. oct. in E + 2 (8) and E + 3 (9) = 28 + 8 = ...		36
Remainder		1.5
Remainder being positive, and being now less than next difference, subtract $\frac{E+3}{6} = \frac{9}{6} =$		1.5

There being no remainder, the given number is a perfect third power of which 9 (the greatest edge which has come into the process) is the required root.

Let it be required to find the greatest third power and remainder over, if any, contained in the given number 963, and also the root of such power.

Given number	963
Divide it by 6. Quotient	160.5
There are two numbers of octahedra in the table which are equally near the quotient, viz.: 35 (E . 6) and 286 (E . 12). Let the less be taken as the nearest number.	
Subtract nearest no. (E . 6)	35
First remainder	125.5
Subtract diff. bet. nos. oct. in E (6) and E + 1 (7), as in table	21
Remainder	104.5

Subtract diff. bet. nos. oct. in E + 1 (7) and E + 2 (8) = 21 + 7 = ...	28
Remainder	76.5
Subtract diff. bet. nos. oct. in E + 2 (8) and E + 3 (9) = 28 + 8 = ...	36
Remainder	40.5
Remainder being positive and less than next difference, subtract $\frac{E+3}{6} = \frac{9}{6} =$ . .	1.5
Remainder	39
Multiply remainder by	6
Product = remainder over, required . .	234
Subtract same from given number . . .	963
Remainder = greatest third power required .	729

$n = 9$.

With the same given number, let the greater of the two numbers in the table, which are equally near the quotient, be taken as the nearest number.

Given number		963
Divide it by 6.	Quotient	160.5
Subtract nearest no. (E . 12) . . .		286
	First remainder	− 125.5
Subtract diff. bet. nos. oct. in E (12) and E − 1 (11), as in table		− 66
	Remainder	− 59.5
Subtract diff. bet. nos. oct. in E − 1 (11) and E − 2 (10) = − 66 − (− 11) = . .		− 55
	Remainder	− 4.5

Remainder still being negative, subtract diff. bet. nos. oct. in E − 2 (10) and E − 3 (9) = − 55 − (− 10) = ...	− 45
Remainder	40.5
Remainder being now positive, and greater than the quotient of the least edge which has come into the process, E − 3, divided by 6, subtract $\frac{E-3}{6} = \frac{9}{6} =$ . . .	1.5
Remainder	39
Multiply remainder by	6
Product = remainder over, required . .	234
Subtract same from given number . .	963
Remainder = greatest third power required	729

$n = 9$.

Let the given number be	1716
Divide it by 6. Quotient	286
Subtract nearest no. oct. in table (E . 12) .	286
First remainder	0
Subtract diff. bet. nos. oct. in E (12) and E − 1 (11), as in table	− 66
Remainder	66
Remainder being positive and greater than $\frac{E-1}{6} = \frac{11}{6}$, subtract $\frac{11}{6} =$ . . .	1.833
Remainder	64.166
Multiply remainder by	6
Given number = $(E-1)^3 = 11^3$ + rem. over	385

It is also equal to $E^3 - E = 12^3 - 12$, being the point at which the first remainder becomes 0.

Again let the given number be . . .		1729
Divide it by 6.	Quotient	288.166
Subtract nearest no. oct. in table (E . 12) .		286
	First remainder	2.166
Remainder being a positive quantity but not exceeding the number shown in the second column of differences in the table, subtract $\frac{E}{6} = \frac{12}{6} =$		2
	Remainder	.166
Multiply remainder by		6
Given number $= E^3 = 12^3 +$ remainder over		1.

E^3 being the point at which subtraction from the first remainder begins with a positive quantity, namely, $\frac{E}{6}$.

Again, let the given number be . . .		1727
Divide it by 6.	Quotient	287.833
Subtract nearest no. oct. in table (E . 12) .		286
	First remainder	1.833
Remainder being positive but less than $\frac{E}{6} = \frac{12}{6}$, subtract diff. bet. nos. oct. in E (12) and E − 1 (11), as in table . .		− 66
	Remainder	67.833
Subtract $\frac{E-1}{6} = \frac{11}{6} =$		1.833
	Remainder	66

Multiply remainder by	6
Given number = $(E-1)^3 = 11^3$ + rem. over	396

It is also equal to $E^3 - 1 = 12^3 - 1$, being the point at which subtraction from the first remainder begins with a negative quantity.

Let it be required to find the edge of a regular tetrahedron of which the volume is . . . 140608

Divide it by 6.	Quotient	23434.666
Subtract nearest no. oct. in table (E . 54)		26235
	First remainder	− 2800.333
Subtract diff. bet. nos. oct. in 54 and 53, as in table		− 1431
	Remainder	− 1369.333
Subtract diff. bet. nos. oct. in 53 and 52 $= -1431 - (-53) =$ . . .		− 1378
	Remainder	8.666
Remainder being now positive and equal to $\frac{52}{6}$, subtract $\frac{52}{6} =$		8.666

$n = 52$.

Let it be required to find the greatest third power and remainder over, if any, contained in the given number 157463, and also the root of such power.

Given number		157463
Divide it by 6.	Quotient	26243.833
Subtract nearest no. oct. in table (E . 54)		26235
	First remainder	8.833

Remainder, although positive, not being equal to $\frac{54}{6}$, subtract diff. bet. nos. oct. in 54 and 53, as in table . .	-1431
Remainder	1439.833
Subtract $\frac{53}{6}=$	8.833
Remainder	1431
Multiply remainder by	6
Product = remainder over, required .	8586
Subtract product from given number .	157463
Rem. = greatest third power required .	148877

$n = 53$.

Again, let the given number be . .	218000000
Divide it by 6. Quotient	36333333.333
Subtract nearest no. oct. (E, 600) .	35999900
First remainder	333433.333
Subtract diff. bet. nos. oct. in 600 and 601	180300
Remainder	153133.333
Subtract $\frac{601}{6}=$	100.166
Remainder	153033.166
Multiply remainder by . . .	6
Product = remainder over, required .	918199
Subtract product from given number	218000000
Rem. = greatest third power required	217081801

$n = 601$.

Thus, with the aid of the table, every exact third root consisting of one, two, three, or four figures, and the first six of five figures, can be readily found. Beyond 10005 it is manifest could also be found by continued subtractions of differences between the numbers of octahedra in E + 3 and E + 4, and between those in E + 4 and E + 5, and so on, but this procedure would soon become very irksome.

To construct a table by means of which every exact third root of five figures and the first few of six could be found, would require that it should be prolonged up to 16667 lines.

But the fifth figure may be reached approximately by means of the table, and afterward exactly by calculation, as follows:

Let it be required to find the third root of the given number 388441751777344

Divide it by 6. Quotient 64740291962890.666

The first four figures of the root will now be found by mere inspection of the table to be 7296. The number of octahedra in a regular tetrahedron of the edge of 72960 may be found therefrom as follows:

No. oct. in 7296 . . .	64729643840
Add $\frac{n}{6} =$	1216
and affix 3 ciphers to the sum .	64729645056000

This is the quotient of S divided by 6 when $n = 72960$.

Subtract $\frac{n}{6} =$	12160
No. oct. in 72960 . . .	64729645043840

Comparing now the difference by which the quotient of the given number divided by 6 exceeds the number of octahedra thus found with the difference between the numbers of octahedra in tetrahedra of edges 72960 and 72961, and observing that the product of the latter multiplied by 4 is nearly equal to the former, it will be evident that if the given number is a perfect third power, the fifth figure of its root will be 4, and the process to demonstrate it may be further continued as follows:

To the number of octahedra found as above, add as follows:	64729645043840
Diff. bet. nos. oct. in 72960 and 72961 $\left(= \frac{72960 \times 72961}{2} = 2661617280\right) \times 4 =$	10646469120
$+ 72961 \times 3 =$	218883
$+ 72962 \times 2 =$	145924
$+ 72963 \times 1 =$	72963
Subtract sum	64740291950730
from quotient of given number divided by 6, as above	64740291962890.666
Remainder	12160.666
Subtract $\frac{72964}{6} =$	12160.666

$n = 72964$.

From the number of octahedra in 72960 and differences between that and the numbers in 72959 and 72961 may be found the exact third roots of all perfect third powers from 72930^3 up to and including 72990^3 and the third roots and last remainders of all intermediate numbers. Beginning with 72930^3 and going back to and including 72871^3, the first four figures of the root would be found by means of the 1215th line of the table, and beginning with 72991^3 and going forward up to and including 73050^3, they would be found by means of the 1217th line. The range is thus limited to 30 in each direction.

But the range, in so far as the calculation required is concerned, is in fact limited to 10 in one direction only. Thus, if the given number in the foregoing illustration had been such that by inspection of the table and comparison of the quotient with the number of octahedra in 7296, in connection with the differences given in the same line, it had been seen that the first four figures were probably 7295 or 7297, then the number in 72950 or 72970 would have been sought instead of that in 72960, and in like manner in respect to 7294 or 7298, &c.

If the given number in the foregoing example had been such that the fifth figure of the root would have been 9, then there would have been 9 multiplications towards the close of the process instead of 4 as in the example. The process in respect to 8 of these multiplications (3 in the example), namely, all except the first, can be shortened and will be found to apply to finding all the required figures of the root. Such shortening of the process in the case under consideration consisted in finding the product of 72960 by the sum of the multipliers 8, 7, 6, 5, 4, 3, 2, 1 (3, 2, 1 in the foregoing example) and adding thereto the term of the following series, the number of which is equal to the number of the required figure of the root, instead of finding successive products by such multipliers as in the example, such product by the sum of the multipliers with the term of the series added thereto being equal to the sum of such successive products. The sum of the multipliers is equal to the quotient by 2 of the product of the highest of such multipliers by itself + 1.

Nos. of terms,	1	2	3	4	5	6	7	8	9,	&c.
Series,	0	1	4	10	20	35	56	84	120,	&c.
1st or. diff's		1	3	6	10	15	21	28	36,	&c.
2d or. diff's.		2	3	4	5	6	7	8,	&c.	

The series was found as follows: The author considered the required fifth figure of the root to be successively 2, 3, 4, 5, 6, 7, 8 and 9, and found in each case (except the first) the sum of the products of the multiplications as in the foregoing example (that is, in each case, of multiplications equal in number to the number of the required figure of the root less 1), namely, in the case of 3, of 72961 × 2 and of 72962 × 1, in the case of 4, as in the foregoing example, and so on up to the case of 9, in which the multiplications began with 72961 × 8 and ended with 72968 × 1. Then dividing 72961 in the case of 2 and the sum of the products in each case thereafter by 72961 the quotients were found to be the sums respectively of the multipliers with remainders constituting the series as above given. The term of the series in each case was then found to be the difference between the sum of the products and the product of 72960 by the sum of the multipliers.

The term of the series required in the process as to any given number of which the third root is sought may be found by means of the term of the first order of differences on the left hand of the required term of the series, and of the term of the second order of differences under such required term, and of the number of such required term.

Let T signify the required term of the series, t the number of the term, a the term of the first order of differences as above, and b the term of the second order. Then

$$b = t$$

$$a = \frac{b^2}{2} - \frac{b}{2}$$

$$T = (b + a) \times \frac{t - 1}{3}.$$

But there is a shorter method. The foregoing series is the series of octahedra contained in regular tetrahedra of edges beginning with 1 and continued according to the series of natural numbers. (See Analysis on page 68.) The required term of the series is therefore equal to the number of octahedra contained in a regular tetrahedron, the edge of which is equal to the number of the term (which, as before stated, is equal to the number of the required figure or figures of the root), and may be found as shown on page 69. Thus,

$$T = t^2 \times \frac{t}{6} - \frac{t}{6}.$$

Let it be required to find the third root of the given number 388521613429209

Quotient by 6 . . . . . 64753602238201.5

The first four figures of the root will now be found by inspection of the table to be 7296, as in the next preceding example (on pages 81 and 82), and the number of octahedra in 72960 may be found as shown in that example, and by comparison of differences also as in that example, the probability that 9 will be the fifth figure of the root will appear. The process will now be further conducted as follows:

Number of octahedra in a regular tetrahedron of edge 72960 as found on page 81	64729645043840
$+ 2661617280$ (see page 82) $\times 9 =$.	23954555520
$+ 72960 \times \frac{8 \times 9}{2} =$	2626560
$+$ 9th term of series $= \left(9 + \frac{9^2}{2} - \frac{9}{2}\right) \times \frac{8}{3}$ or $9^2 \times \frac{9}{6} - \frac{9}{6} =$	120
Subtract sum	64753602226040
from quotient by 6 of given number .	64753602238201.5
Remainder	12161.5
Subtract $\frac{72969}{6} =$	12161.5

$n = 72969.$

Let the given number of which the third root is required be . . . 1000660148211968148200660001

Quotient by 6 . 166776691368661358033443333.5

The first five figures of the root will be found by inspection of the table to be 10002.

No. of oct. in 10002	166766685001
Add $\frac{n}{6} =$ . .	1667
and affix 15 ciphers . .	1667666866680000000000000000

This is the quotient of S divided by 6 when $n = 1000200000$.

Subtract $\frac{n}{6} =$.	166700000
No. of oct. in 1000200000 . .	1667666866679999998333000000

Diff. bet. nos. oct. in. 1000200000 and 1000200001 =

$$\frac{1000200000 \times 1000200001}{2} = 500200020500100000.$$

Comparing differences as before, the number of the five required figures of the root will be found to be 20001.

No. oct. in 1000200000 .	1667666866679999998333000000
+ diff. as above × 20001 =	10004500610022500100000
+ 1000200000 × $\frac{20000 \times 20001}{2} =$.	200050002000000000
+ 20001st term of series = $20001^2 \times \frac{20001}{6} - \frac{20001}{6} =$	1333533340000
Subtract sum . .	1667766913686613578667400000
from quo. of giv. no. by 6	1667766913686613580334433333.5
Remainder	166703333.5
Subtract $\frac{1000220001}{6} =$	166703333.5

$n = 1000220001.$

It will now be manifest upon due consideration of the foregoing examples, that what can be done by means of the first five or four figures of the root found by the table could as well have been done by means of the first three or two figures found by shorter tables, or by means of only the first figure found directly from the given number and without the aid of any table.

Let it be required to find the third root of the given number 606569944625.

By the usual method of pointing off the figures of the given number it will be seen that the first figure of the root is 8.

The number of octahedra in a regular tetrahedron of edge $8 = 8^2 \times \frac{8}{6} - \frac{8}{6} = 84$.

No. oct. in 8	84
Add $\frac{8}{6} =$	1.333
and affix 9 3's	85333333333.333
removing the decimal point 9 places to the right.	
This is the quotient of S divided by 6, when $n = 8000$.	
Subtract $\frac{n}{6} =$	1333.333
No. oct. in 8000	85333332000

Diff. bet. nos. oct. in 8000 and 8001 $= \frac{8000 \times 8001}{2}$ $= 32004000$.

By comparing differences as before the number of the three required figures of the root will be found to be 465.

No. oct. in 8000	85333332000
$+$ diff. as above $\times 465 =$ . . .	14881860000
$+ 8000 \times \frac{464 \times 465}{2} =$	863040000
$+$ 465th term of series $=$	
$465^2 \times \frac{465}{6} - \frac{465}{6} =$	16757360
Subtract sum	101094989360
from quo. of given no. by 6 . . .	101094990770.833
Remainder	1410.833
Subtract $\frac{8465}{6} =$	1410.833

$n = 8465$.

But there is still another and a much simpler process. To the first figure of the root, as found from the given number, affix as many ciphers as with the first figure will make up the number of places of figures in the required root, as indicated by the pointing off of the figures of the given number, and assuming the number thus formed to be the number of the edge of a regular tetrahedron, find the number of octahedra contained in such regular tetrahedron. Then from the quotient of the given number divided by 6 subtract the number of octahedra thus found, and set down the remainder. Find then the difference between the number of octahedra so found and the number contained in a regular tetrahedron of the next succeeding edge, and divide the remainder found as above by such difference.

The quotient as a whole number (disregarding fractions) resulting from such division, if it shall consist of places of figures equal in number to the number of places of the required figures of the root, will be such required figures; but if it be less, then prefix thereto one or more ciphers, as may be necessary to make up such number of places, and such cipher or ciphers with the quotient will be such required figures.

Find now the number of octahedra contained in a regular tetrahedron of edge equal to the whole root as thus found, and multiply such number of octahedra by 6, and to the product add the number of the root.

The sum will be equal to the given number if the latter be a perfect third power; but if such sum be less, then subtract it from the given number and the remainder will be the excess of the given number over and above the greatest third power therein contained.

To illustrate.—Let it be required to find the third root of the greatest third power and remainder over, if any, contained in the given number 1006012009.

By pointing off the figures of the given number, the first figure of the root is found to be 1, and the number of places of required figures 3.

The number of octahedra contained in a regular tetrahedron of edge 1000 is equal to $1000^2 \times \frac{1000}{6} - \frac{1000}{6} =$ 166666500

Subtract same from quotient of given no. by 6 . . . . 167668668.166

Remainder 1002168.166

Diff. bet. nos. oct. in 1000 and 1001 $= \frac{1000 \times 1001}{2} = 500500$.

$$\frac{1002168.166}{500500} = 2. +$$

Prefixing two ciphers to this quotient, the required root is found to be 1002.

No. oct. in 1002 = $1002^2 \times \frac{1002}{6} - \frac{1002}{6}$ =	167668501
×	6
=	1006011006
+ number of root	1002
Subtract sum	1006012008
from given number	1006012009
Given no. = 1002^3 + remainder . .	1

The table having been thus found to be entirely unnecessary for the purpose of evolution to the third degree, it will probably be asked, Why was it introduced at all, and why the long description and illustrations following it? The answer is, that the shortened processes were not found by the author until after the whole book was in type and nearly ready for the press, awaiting only some final corrections of a few of the plates. The author then found himself in a quandary as to what course he should pursue, whether to strike out the table and all following it down to the end of the second paragraph on page 83 (which would have involved the necessity of striking out also all the following pages down to § 17, on page 107) and write a description of the process as finally reached, and there leave the subject, or to open the book and take in the description of the shortened processes to their ultimate result (striking out, however, a cumbrous method of finding the fifth and subsequent figures of the root by means of further tables, one for each figure), and accompany it with this explanation.

He adopted the latter alternative for the reason that the book as it now stands shows the entire course of investigation by which he reached the final result, and without which, perhaps, such result would never have been attained, and for the further reason that, as with the aid of the table the operations of involution to the third degree and of both evolution and involution to the second degree may be performed (as hereinafter shown) with much greater facility than by the usual arithmetical processes, so it may be, that the table may be found of service in respect to other operations of which the author has no knowledge.

The third power of any number up to 10002, and a few numbers beyond, may be found with the aid of the table by a shorter process than by taking the number three times as a factor.

Thus, let it be required to find the third power, or volume of a regular tetrahedron of the edge of 68.

The nearest edge in the table is 66.

No. oct. in 66, as in table	47905
+ diff. bet. 66 and 67, as in 2d col. diff's .	2211
+ diff. bet. 67 and 68 (= 2211 + 67) . .	2278
=	52394
×	6
=	314364
+ given number	68
=	314432

$= 68^3$.

To find the third power of 4757.

Nearest edge in table 4758.

No. oct. in 4758	17952380459
+ diff. bet. 4758 and 4757, as in 1st col. diff's	− 11316903
=	17941063556
×	6
=	107646381336
+ given number	4757
=	107646386093

$= 4757^3$.

If the given number should be higher than a few beyond 10002, or be of six or more figures, a process could undoubtedly be devised to find the power. The author has not attempted to find such process.

By means of the foregoing table may also be found the second root of all perfect second powers and the root of the greatest second power and remainder over, contained in all intermediate numbers when such root consists of one, two, three, or four figures, or is one of the first six of five figures.

The sum of the differences between the number of octahedra in a regular tetrahedron of any given edge and those of the next preceding and succeeding edges respectively, considered both as positive quantities, is equal to the second power of the number of the given edge.

$$\underbrace{\frac{n(n-1)}{2}}_{\textit{1st diff.}} + \underbrace{\frac{n(n+1)}{2}}_{\textit{2d diff.}} = n^2.$$

To find the second root of any given perfect second power, or such root of the greatest second power and remainder over, contained in any given number, divide the given number by 2, and look in the first column of differences in the table for the number nearest the quotient. If there are two numbers in the first column of differences equally near the quotient, either may be taken. This can occur when the given number is an exact second power only once, namely, in the case of 9^2.

Note the edge (root) in the same line with the nearest number taken.

Compare the nearest number found in the first column of differences with the quotient, and if it be less, observe whether the excess of the quotient is equal to $\frac{E}{2}$ and is not greater than $\frac{3\,E}{2}$.

If such excess is within these limits (both inclusive), then the numbers given in the two columns of differences in the table are the differences sought, the sum of which is equal to the given perfect second power, or to the greatest second power contained in the given number, and E is the required root.

If such excess is less than $\frac{E}{2}$, or if the nearest number exceeds the quotient, then set down the nearest number so found and subtract from it $E-1$. If the remainder exceeds the quotient, subtract therefrom $E-2$, and if, as thus diminished, it still exceeds the quotient, subtract from it $E-3$. The quotient will now exceed the remainder, and the latter will be the first difference sought, and the next preceding remainder, or if there shall have been but one subtraction, then the number taken from the table will be the second difference sought, and the last number subtracted (E − 3 or 2 or 1, as the case may be) will be the required root.

If the excess of the quotient over the nearest number is greater than $\frac{3\,E}{2}$, then go to the second column of differences, same line, and set down the number therein found (which is the second difference in E and first in E + 1) and add thereto E + 1. If the sum be less than the quotient add to it E + 2, and if as thus increased it be still less, add to it E + 3. The sum will now exceed the quotient and will be the second difference sought, and the next preceding sum, or if there shall have been but one addition, then the number taken from the table will be the first difference sought, and the last number added (E + 3 or 2 or 1, as the case may be) will be the required root.

Find in either case the sum of the two differences and subtract it from the given number. If there be no remainder, the given number is a perfect second power; but if there be, then such remainder is the excess of the given number over the greatest second power therein contained, and will be the last remainder that would be found in the usual arithmetical process of extracting the second root.

To illustrate. Let the given number of which the second root is required be 36

Divide it by 2. Quotient, 18.

Nearest no. in 1st. col. diff's in table is 15 (E . 6). Excess of quotient over same being equal to $\frac{E}{2}$, the two differences in the table are the differences sought.

1st diff. in E (6)	15	
2d diff. in E (6)	21	
Find sum of diff's		36
and subtract same from given number		
There being no remainder, ...		..

the given number is an exact second power, of which

E = 6 is the required root.

Again, let the given number be . . .		48

Divide it by 2. Quotient, 24.

Nearest no. in 1st col. diff's in table is 15 (E . 6). Excess of quotient over same not exceeding $\frac{3E}{2}$, the two differences in the table are the differences sought.

1st diff. in E (6)	15	
2d diff. in E (6)	21	
Find sum of diff's		36
and subtract same from given number.		
Given number = $E^2 = 6^2$ + remainder .		12

Again, let the given number be . . .		81

Divide it by 2. Quotient, 40.5.

Two numbers in the first column of differences are equally near the quotient, viz., 15 (E . 6) and 66 (E . 12). Let the latter be first taken as the nearest number.

Nearest number (E . 12)	66	
Same exceeding quotient, subtract E − 1 =	11	
Remainder = 1st diff. in E − 1 (11) . .	55	
Same exceeding quotient, subtract E − 2 =	10	
Remainder = 1st diff. in E − 2 (10) . .	45	
Same still exceeding quotient, sub. E − 3 =	9	
Remainder = 1st diff. in E − 3 (9) . .	36	
Remainder being now less than quotient, bring down next preceding remainder = 2d diff. in E − 3 (9)	45	
Find sum of diff's		81
and subtract same from given number .		..

E − 3 = 9 is the required root.

With the same given number, let the first of the two numbers which are equally near the quotient be now taken as the nearest number.

Given number		81
Divide it by 2. Quotient, 40.5.		
Nearest no. (E . 6) 15. Excess of quotient over same being greater than $\frac{3E}{2}$, take no. in 2d col. diff's	21	
Add E + 1 = ...	7	
Sum = 2d diff. in E + 1 (7)	28	
Same being less than quotient, add E + 2 =	8	
Sum = 2d diff. in E + 2 (8)	36	
Same being still less than quotient, add E + 3 = ...	9	
Sum = 2d diff. in E + 3 (9)	45	

Sum being now greater than quotient, bring down next preceding sum = 1st diff. in E + 3 (9) 36

Find sum of diff's 81

and subtract same from given number . . .

E + 3 = 9 is the required root.

Again, let the given number be 22648080

Quotient by 2. 11324040.

Nearest no. in 1st. col. diff's (E . 4758) 11316903. Excess of quotient over this number not exceeding $\frac{3\ E}{2}$, the diff's in table are those sought.

1st diff. in 4758 11316903

2d diff. in 4758 11321661

Find their sum 22638564

and subtract same from given no.

Given no. = 4758^2 + remainder 9516

Again let the given number be 22648082

Quotient by 2. 11324041.

Nearest no. in 1st col. diff's (E . 4758) 11316903. Excess of quotient over this number being greater than $\frac{3\ E}{2}$, take no. in 2d col. 11321661

Add E + 1 = ... 4759

Sum = 2d diff. in 4759 . . . 11326420

Bring down no. taken from table = 1st diff. in 4759 . . . 11321661

Find sum of diff's 22648081
and subtract same from given no.

Given no. $= 4759^2 +$ remainder . . . 1

The fifth and subsequent figures of the second root could probably be found by a process analogous to that for finding the fifth and subsequent figures of the third root. The author has made no attempt to find such process.

The second power of any number given in the last column of the table is the sum of the differences in the same line considered both as positive quantities. The second power of any number intermediate between any two consecutive numbers in the last column of the table may be found with the aid of the table, as follows:

Let it be required to find the second power of 598 intermediate between 594 and 600, the nearest number being 600.

No. in 1st col. diff's in 600 179700
— 599
= (2d diff. in 598) 179101
+ (179101
— 598 = 1st diff. in 598 =) . . . 178503
= 357604
$= 598^2$.

Again, let the given number be 7299 intermediate between and equally near 7296 and 7302 in the last column of the table. Take 7296.

No. in 2d col. diff's in 7296	26619456
+	7297
+	7298
= (1st diff. in 7299)	26634051
+ (26634051 + 7299 = 2d diff. in 7299 =) .	26641350
=	53275401
$= 7299^2$.	

In the case of higher numbers than those to which the table is applicable, resort would of course be necessary to a further process which could probably be devised for the purpose. As before, the author has made no attempt to find such process.

It may perhaps be said that the operations of evolution and involution are performed by means of logarithms which apply as well to finding higher powers and their roots, and that the foregoing processes apply only up to the third degree, and that as in respect to such higher powers there can be no physical representation, so there can be no tables constructed nor processes devised analogous to the foregoing table and processes for the purpose of finding them or their roots. But the foregoing table and processes are not given as a proposed substitute for the table of, and processes by, logarithms, in so far as they apply, but as reasons and illustrations in support of an affirmative answer to the main question propounded in this book. They could never have been discovered by an analysis of the cube.

In the description of the method of finding the third root by means of the table, it is stated that it can never occur when the given number is an exact third power, that there will be two numbers of octahedra in the table equally near the quotient of the given number divided by 6.

The series of numbers in respect to such quotient of each of which there are in the table two numbers of octahedra equally near, may be said to begin with 105, the quotient of which divided by 6 is 17.5, to which 0 and 35 (the first number of octahedra in the table) are equally near, and in respect to this number there is this peculiarity, which will not again occur in the forward direction of the series. The greatest exact third power contained in 105 is $64 = 4^3$, and the remainder over is 41. But 105 is also equal to $3^3 + 78$, and must be considered as such with reference to the series, which is as follows:

$$
\begin{array}{rcrcr}
105 & = & 3^3 & + & 78 \\
963 & = & 9^3 & + & 234 \\
3765 & = & 15^3 & + & 390 \\
9807 & = & 21^3 & + & 546 \\
20385 & = & 27^3 & + & 702 \\
* & & * & & * \\
806775 & = & 93^3 & + & 2418 \\
972873 & = & 99^3 & + & 2574 \\
1160355 & = & 105^3 & + & 2730 \\
1370517 & = & 111^3 & + & 2886
\end{array}
$$

and so on.

The term of this series lying between any two consecutive numbers of octahedra in the table may be found by adding together such numbers and multiplying their sum by 3.

Thus, $(35 + 286) \times 3 = 963$.

All the terms of the third order of differences of the series are equal, each being $1296 = 6^4$.

The difference between the roots of each two consecutive greatest third powers (the first being considered such as above stated) is in all cases 6, and that between each two consecutive remainders over, is in all cases 156. The remainder over, in the first term of the series, it will be observed, is one-half such difference.

This difference, 156, is made up of equal differences between each two consecutive terms of an expanded series intermediate between any two consecutive terms of the above series, as will be seen by the following analysis of the expanded series intermediate between the first and second terms, to be considered in connection with the first two lines of the table expanded, as subsequently shown.

$$105 = 3^3 + 78$$
$$168 = 4^3 + 104$$
$$255 = 5^3 + 130$$
$$372 = 6^3 + 156$$
$$525 = 7^3 + 182$$
$$720 = 8^3 + 208$$
$$963 = 9^3 + 234$$

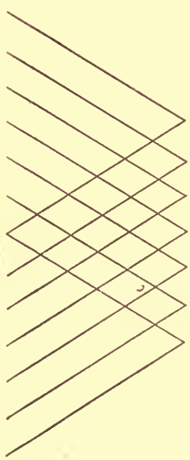
All the terms of the third order of differences of the expanded series are equal, each being 6, and the difference between any two consecutive remainders over, in the last column is in each case 26.

The following table exhibits the first two lines of the table of octahedra fully expanded, the last column showing the edges from 1 up to 12 instead of in multiples of 6, as in the original table, but with the columns of differences omitted to give place to a diagram showing two numbers of octahedra, six lines apart, equally near the quotient of each term of the foregoing expanded series divided by 6.

TABLE OF OCTAHEDRA.

(*Expanded.*)

Value of $\frac{n}{6}$.	Nos. of Octahedra.		Edge n.
0	0		0
.166	0		1
.333	1		2
.5	4	17.5 = 105 ÷ 6	3
.666	10	28 = 168 ÷ 6	4
.833	20	42.5 = 255 ÷ 6	5
1.	35	62 = 372 ÷ 6	6
1.166	56	87.5 = 525 ÷ 6	7
1.333	84	120 = 720 ÷ 6	8
1.5	120	160.5 = 963 ÷ 6	9
1.666	165		10
1.833	220		11
2.	286		12



A continuation of the expanded series backward will show that its first term is 27 and the first remainder over, 26. Thus,

$$
\begin{array}{rcccc}
0 & & & & \\
27 & = & 1^3 & + & 26 \\
60 & = & 2^3 & + & 52 \\
105 & = & 3^3 & + & 78
\end{array}
$$

The number 26 has been hereinbefore (page 53) shown to be the base of the differences between the volumes of figures of any two consecutive orders in the table of natural involution.

It is also the apparent volume of a tetrahedron of edge 3 as calculated from an external view as hereinbefore described (page 66), the concealed tetrahedron being the central figure, and is the difference between the volume of the tetrahedron of the first order and that of one of the second order in the suggested table based upon the tetrahedron as the central figure (page 56) and the base of all subsequent differences between the volumes in any two orders in the same table.

In the description of the method of finding the second root by means of the table, it is stated that it can occur but once when the given number is an exact second power, namely, in the case of 9^2, that there will be two numbers in the first column of differences in the table equally near the quotient of the given number divided by 2.

The series of numbers in respect to the quotient of each of which divided by 2, there are two numbers in the first column of differences in the table equally near, is as follows:

$$15 = 3^2 + 6$$
$$81 = 9^2$$
$$219 = 15^2 - 6$$
$$429 = 21^2 - 12$$
$$711 = 27^2 - 18$$

and so on.

The term of this series lying between any two consecutive numbers in the first column of differences in the table is the sum of such numbers.

Thus, $15 + 66 = 81$.

All the terms of the second order of differences of the above series are equal, each being 72, and the difference between any two remainders over in the last column is in each case 6.

The expanded series intermediate between the first two terms of the foregoing series and continued beyond, is as follows:

$$
\begin{aligned}
15 &= 3^2 + 6 \\
21 &= 4^2 + 5 \\
29 &= 5^2 + 4 \\
39 &= 6^2 + 3 \\
51 &= 7^2 + 2 \\
65 &= 8^2 + 1 \\
81 &= 9^2 \\
99 &= 10^2 - 1 \\
119 &= 11^2 - 2 \\
141 &= 12^2 - 3
\end{aligned}
$$

and so on.

All the terms of the second order of differences of the expanded series are equal, each being 2, and the difference between any two remainders over, in the last column is in each case 1.

The author notes the following further observations in the analysis of the several courses of a regular tetrahedron and of a series of whole figures, as shown on page 68.

The figures in the units' places in the first five, ten, or twenty lines of the table (considered as continued) are as follows:

In respect to Courses.

Of tetrahedra.....1.3.7.3.1.

Of octahedra......0.1.3.6.0.5.1.8.6.5.5.6.8.1.5.0.6.3.1.0.

Of vols. of oct.....0.4.2.4.0.

Of total vols......1.7.9.7.1.

In respect to Whole Figures.

Of tetrahedra.....1.4.1.4.5.6.9.6.9.0.

Of octahedra......0.1.4.0.0.5.6.4.0.5.0.6.4.5.0.0.6.9.0.0.

Of vols. of oct.....0.4.6.0.0.

Of total vols......1.8.7.4.5.6.3.2.9.0.

and they are the same in each succeeding five, ten, or twenty lines, as the case may be. The sequence of the figures in respect to the courses read backward is the same as when read forward.

The numbers of the tetrahedra and octahedra appearing on the outer faces of the several courses (see pages 65 and 66) constitute two like series, but beginning with course 1 as to the tetrahedra and with course 2 as to the octahedra. The terms of these series in any course are equal to $n\left(\frac{n}{2}+.5\right)$ as to the tetrahedra and to $n\left(\frac{n}{2}-.5\right)$ as to the octahedra, and such terms are the differences between the numbers of octahedra contained in the whole figure of like edge as the course and in whole figures of the next succeeding and preceding edges.

In the second order of differences of the two series of numbers of tetrahedra and octahedra respectively contained in the several courses, and in the third order of differences of the two like series contained in whole figures, the terms become equal, and are in each case 2 in respect to the series of tetrahedra and 1 in respect to those of octahedra.

The terms of the third order of differences of the series of numbers of octahedra in the table on page 71 become equal, and are in each case $216 = 6^3$.

§ 17. If the octahedron be revolved one-fourth of a revolution on each of its three axes, its moving points will describe three circles, each bisecting the other two. If such circles are delineated on the surface of the circumscribed sphere of the octahedron, they will be great circles of the sphere, and will divide such surface into eight equal trirectangular spherical triangles, representing the primary division of the surface of the sphere as delineated by geometers, geographers, and astronomers.

The further division of the surface of the sphere is made by geographers and astronomers by great circles representing meridional lines by which longitude is reckoned, and small circles parallel to the equator by which latitude is reckoned. These artificial circles with the three primary natural ones by their intersections divide the surface of the sphere into quadrilaterals (varying in number according to the number of the artificial circles), except immediately about the poles, where the division is into isosceles triangles.

These quadrilaterals are all irregular, and increase in the dimensions of two of their sides as they approach the equator from either pole, and the degrees of longitude as they are measured on different parallels of latitude on the surface of the earth, considered as the sphere, vary in length from sixty geographical miles on the equator to zero at the poles.

Let the circumscribed sphere of the octahedron, having delineated thereon only the three primary great circles, be considered as having the points of intersection of such circles (which are the points of the inscribed octahedron) designated by the logical symbols successively in the three different ways shown in the three diagrams on pages 40, 41, and 42.

In each case the two complete processes of reasoning (complete in the logical sense but not in their combinations on the faces of the octahedron, or the surface of its circumscribed sphere), have their terms of beginning (of the progressive sorites in each case) at the poles respectively, but the ultimate terms (of both the progressive and regressive sorites) in respect to each process will be found as follows:

In each of the first and second cases, that of the progressive sorites at one point and that of the regressive at another, both on the equatorial line, the designations, in the first case, being on the opposite side of the line from the point of beginning, but, in the second case, on the same side as the point of beginning.

In the third case, that of the progressive sorites at a point on the equatorial line, with the designation on the same side as the point of beginning, and that of the regressive sorites at the opposite pole from the point of beginning.

In the first case, no sorites has its two syllogisms on adjacent faces; in the second, one sorites of each process only, and in the third both; but in the third case, the two sorites of each of the processes are not combined in regular order respectively, and procedure cannot be made from one to the other, and the two processes are not, as it would seem they should be, on hemispheres bounded on the surface by the equatorial line, but are on hemispheres bounded by a great circle passing through the poles.

§ 18. If the oct·astron be revolved one-third of a revolution on each of its four axes, its moving points will describe eight circles. If such circles are delineated upon the surface of the circumscribed sphere of the oct·astron, each two parallel ones described by each partial revolution will divide such surface into three zones, and the planes of each such two parallel circles will trisect the related axis of the sphere. The eight circles, by their mutual intersections, divide the surface of the sphere into forty-two figures, of which eighteen are quadrilaterals, and twenty-four isosceles triangles. Twelve of the eighteen quadrilaterals are rhombs and six, squares.

All the figures of the same kind are equal to each other.

Let now such a sphere with circles so delineated be considered as in hand, and let those points on the surface, which are the points of the included oct·astron, be marked with the symbols of the two complete processes of reasoning, as the points of the oct·astron are marked as shown in figures 17 and 18, on page 34, and in the description following those figures, and let the axis, the extremities of which are marked X X X and N′ N′ N′, be regarded as vertical and as the axis of revolution of the sphere.

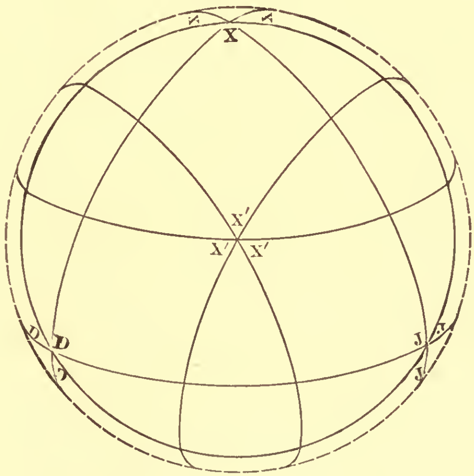
Each point will be found designated by but one symbol instead of by two or four, as in the case of the circumscribed sphere of the octahedron, and the two complete processes of reasoning, beginning at the poles respectively, instead of being disjoined or imperfectly conjoined as in such case, will be found, the two sorites of each process perfectly combined, and the two complete processes perfectly conjoined, overlapping each other and having their ultimate terms, both progressive and regressive of each process, at the same point, but of the two processes considered relatively to each other at opposite points of a diameter of the sphere, but on the lines of two different circles, the bases of the middle zone ; the term of beginning of each process at the poles respectively being related to the ultimate term on the circle farthest removed from such pole.

There will be no equatorial line. Three great circles may be drawn dividing each of the rhombs into two equal regular triangles, and each of the squares into four equal right-angled triangles. These great circles would be the boundaries of imaginary planes passing through the centre of the sphere, in each of which planes two axes and four edges of the octahedron included in the included oct·astron would lie, but neither of the circles would be equatorial relatively to the axis of revolution of the sphere, or to either of the other axes of the included oct·astron.

These great circles would, however, probably never be required. To lay them down and thereby draw a diagonal through each of the rhombs would be equivalent to expressing the unexpressed conclusion of the first, which is the unexpressed premise of the second of the series of two syllogisms into which a simple sorites may be expanded. A simple sorites is as manifestly conclusive on its face as a simple syllogism.

Let the reader now consider that the sphere which he has in hand is the circumscribed sphere of an oct·astron of the edge of 1, and that it is held with its axis, X N′, vertical, in such position below the eye that the great triangle X J D is in full view. The following figure will then be presented :

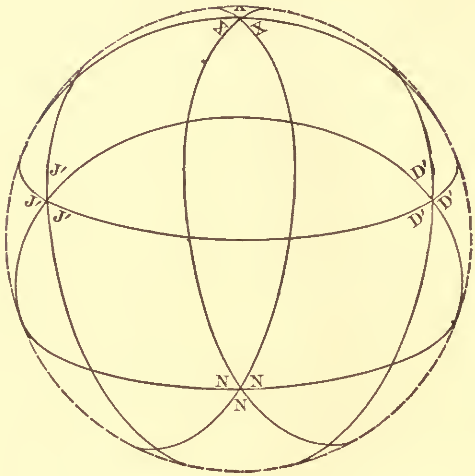
Fig. 34



One-third only of the surface of the sphere is plainly visible, viz.: the zone about the point X′ and bounded by that one nearest such point of the two circles described by the moving points of the oct·astron in the partial revolution of the sphere about the axis X′ N.

Revolving the sphere one-half of a revolution, the following figure will present itself :

Fig 35



If now the sphere be lifted up to a position as far above as it was previously held below the eye, the great triangle N′ D′ J′ will come in full view, and upon turning it about, one-half of a revolution (or one-sixth or five-sixths), the figure presented will be similar (except as to symbols) to the foregoing, but with the two fully shown intersecting figures in the external form of lunes (and which will be herein called lunes) on the middle and lower instead of the middle and upper zones of the sphere.

By revolving the sphere, as held below the eye from left to right, the great triangles X D N and X J N will successively come in view, and as held above the eye, triangles N′ J′ X′ and N′ D′ X′; and by turning the figure so that the vertical axis X N′ shall be horizontal, with the pole N′ toward the eye, triangle N D J will come in view, and with the pole X toward the eye, triangle X′ J′ D′.

The figures in the form of lunes are directly over the edges of the included oct·astron and separate the great triangles of each process from each other. Each great triangle is bounded by three lunes, and the outer line of each lune (with reference to any great triangle), upon being revolved on its chord, will coincide with the surface of the sphere until it shall reach and coincide with the inner line, the side of the great triangle.

Each rhomb is common to two of the great triangles, viz.: one of each of the two processes, descending and ascending, and the points of their acute angles only are designated, one by a symbol of one process and the other by a symbol of the other, each of the extremes of the process in either direction being connected by three rhombs with the opposite extremes respectively and the two middle terms of the process in the other direction.

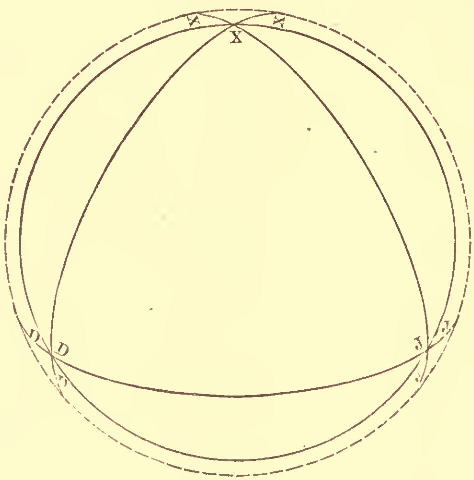
The zone about either pole of any axis consists of three rhombs, nine small triangles, and three squares, and the middle zone consists of six rhombs and six small triangles. The area of each of the rhombs is therefore equal to the sum of the areas of one of the small triangles and one of the squares.

If the lines of but one of the complete processes of reasoning, descending or ascending throughout, were delineated upon the surface of the sphere, the figures thereby produced would consist of four great triangles and six lunes dividing the entire surface of the sphere. Such lines would be delineated by the moving points of a regular tetrahedron of the edge of 2, in its partial revolutions, as before described in respect to the oct·astron, the circumscribed sphere of such a tetrahedron being equal to that of an oct·astron of the edge of 1.

There would be but four circles and four points of intersection of their lines, each point designated by one of the symbols of the logical process. Each point would be polar to one circle only, and there would be no opposite poles.

The following figure exhibits a zone of the sphere with circles so delineated.

Fig. 36



The zone consists of a great triangle bounded by three lunes, its area being one-third that of the surface of the sphere. The great triangle with one-half of each of the lunes by which it is bounded is equal in area to one-fourth the surface of the sphere.

§ 19. If partial sections be made into the body of the sphere on which all the circles are delineated, following the planes of the circles down to the chords of the arcs which form the sides of the figures on the surface, and the spherical surfaces be cut off from each figure, the resulting solid figure will be one of surpassing beauty and symmetry, the faces of which will be the underlying planes of the figures on the surface of the sphere.

The dimensions of the parts of each of the faces are as follows: The sides of the rhombs are each .866, equal to the altitude of the regular triangle (side 1); their acute angles are 70° 31′ 42″, equal to the dihedral angle of a regular tetrahedron, and their obtuse angles are 109° 28′ 18″, equal to the dihedral angle of a regular octahedron. If a diagonal were drawn bisecting the obtuse angles, its length would be 1. The base of each of the small triangles is .5, the other sides are each .866, the angle opposite the base 33° 33′ 30″, and the other angles each 73° 13′ 15″. The squares have their sides each .5, and their angles, of course, each 90°.

§ 20. The polygons and angles on the surface of the circumscribed sphere of the oct·astron are not spherical polygons and angles as defined by geometers, which are all bounded by arcs of great circles, and all have relation to the centre of the sphere. But the angles of the great triangles are the dihedral angles of the planes of the circles by which they are formed, in like manner as the angles formed by the intersection of two great circles are the dihedral angles of the planes of such circles, and the obtuse angles of the rhombs are also such dihedral angles, and all such angles correspond to those of the underlying planes of the rhombs.

In the case of the small triangles, the angles of the underlying planes do not correspond to the dihedral angles of the planes of the circles by which they are formed. But by consideration of the preceding figures, it will be manifest that the small triangles are not to be considered by themselves, but each two with their intervening square as constituting a lune to be considered as a whole in any process in which it may be involved, the intersecting lines by which the lune is divided, being part of a wholly distinct configuration, and not to be taken into consideration in the process. The square is thus entirely eliminated from the figures.

§ 21. If now the partial sections into the body of the sphere be continued and made complete along the planes of all the circles, the figure described in § 19 will be divided into fifty-one parts corresponding to the number of figures in an octahedron of the second order, viz.: the nine perfect and regular figures of the included oct·astron of the first order and parts, viz.: one-half of each of the twelve octahedra first mentioned, one-fourth of each of the twenty-four tetrahedra secondly mentioned, and one-eighth of each of the six octahedra thirdly mentioned in the description of the construction of the octahedron of the second order hereinbefore contained (page 51).

The twelve octahedra first mentioned will have been divided each by a plane passing from one point to the opposite point through four faces adjacent in pairs and opposite in pairs, bisecting such faces and forming the rhombs ; the twenty-four tetrahedra secondly mentioned will have been divided each by a plane passing from one of its points to the middle points of two sides of its opposite face, and forming the small triangles, and the six octahedra thirdly mentioned will have been divided each by a plane passing through four of its faces having a common point, beginning at the middle point of one edge of the octahedron, and cutting such faces in lines parallel to their sides opposite such common point, and forming the squares.

It has been hereinbefore stated (on page 56) that the tetrahedra in an octahedron of the second order are analogues of compound logical processes through which the two complete simple processes on the faces of the oct·astron of the first order are brought into perfect union. Such union consists in establishing the relation between like extremes of the two processes. The symbols of such extremes as ultimately reached through such compound processes do not, however, designate opposite poles of two axes of the oct·astron considered as consisting of two intervolved tetrahedra, but are like symbols of the extremes of the processes considered as conducted on the faces of two of the superposed tetrahedra (as described in § 11, on page 37), one in each direction, and designate points of such tetrahedra which fall upon the octahedron included in the oct·astron. Such points do not come to the surface of the sphere, but the symbols designating them may be considered as brought to such surface on the faces of the rhombs at their obtuse angles, and by combinations of the ultimate results of the two processes considered as conducted upon the tetrahedra as superposed and so brought to the surface with the conclusions of the two processes considered as conducted upon the tetrahedra as intervolved, the relation may be established between like extremes of the two processes, the symbols of which designate opposite poles of each of two axes of the oct·astron and of its circumscribed sphere. All which will hereinafter (in the appendix) be fully shown.

The circumscribed sphere of a tetrahedron or of an oct·astron of the first order, and the inscribed sphere of an octahedron of the second order, are equal to each other, and the three considered as contained in an octahedron of the second order are identical, and may be regarded as the emblem of triniunity.

The author concludes this, the main part of his treatise, with the following question:

Is not the delineation of the surface of the sphere, produced by the revolution of the octahedron on its three axes and supplemented by artificial lines, the better adapted for the description of the terrestrial sphere for all the ordinary purposes of life; and is not that with natural lines only, produced by the revolution of the oct·astron on its four axes, the better adapted for all scientific purposes, and especially with reference to the celestial sphere?

APPENDIX.

THE following illustrations of the analogy between compound logical processes and combinations of simple geometrical figures could not have been introduced in the foregoing treatise without breaking its continuity. They are therefore given in the form of an appendix, but to be considered as a part of the treatise.

The typical simple sorites of Concrete Logic is one in which the magnus term is an individual thing which can be predicated of nothing (except itself), the maximus term, the highest term that can be predicated of the magnus term, but of which nothing (except itself) can be predicated, and the major-middle and minor-middle terms, a genus and species respectively, of the first of which the maximus term may be predicated, and the second of which may be predicated of the magnus term, and which are proximate to each other, so that the truth of the proposition in which they are compared (the middle premise) is readily recognized and admitted. (The foregoing description is in the ascending direction, but, by changing the expressions, it may be made applicable also to the descending.)

In such a sorites there can be no additional terms introduced except those which are subsidiary, elucidating either the relation between the major-middle and maximus terms, or that between the magnus and minor-middle terms, or, by different new terms, both.

This will clearly appear (but by symbols indefinite in material signification, of which four are assumed to be as above described) by the following illustrations, in which figures 1 and 2 are reproduced, but with the ultimate point in each represented as inaccessible only in a direct line from the point of beginning, but visible from such point, and as incapable of being either seen or reached directly from the third point by reason of an obstruction in each case, but capable of being reached indirectly by way of the new point introduced in each case.

Fig. 1 a

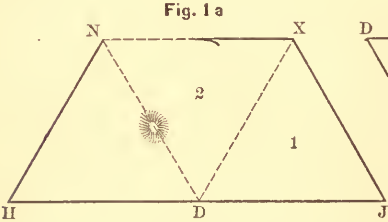
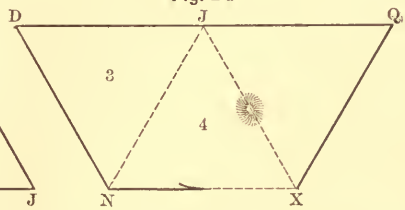


Fig. 2 a



The sorites are now compound, and fully expressed are as follows:

In the descending direction.

X comprehends J,
J comprehends D;
– – – – – – – –
D comprehends H,
H comprehends N;
∴ – – – – – – – –
D comprehends N,
and ∴ X comprehends N.

In the ascending direction.

N is comprehended in D,
D is comprehended in J;
– – – – – – – – – – –
J is comprehended in Q,
Q is comprehended in X;
∴ – – – – – – – – – – –
J is comprehended in X,
and ∴ N is comprehended in X.

But a simple sorites may have as its maximus term a subaltern genus, and as its magnus term a subaltern species relatively to new terms which are higher genera or lower species respectively, and which may be found by investigation in either or each direction and brought into the reasoning process in opposite directions respectively, and in such case the new terms, if brought in in both directions, will supplant the original maximus and magnus terms (or if in one direction only, then either, as the case may be), and the two latter will become major-middle or minor-middle terms, or subsidiary middle terms, and the logical significations of the original middle terms will be changed, the major-middle becoming minor-middle and the minor-middle subsidiary in the descending direction, and the minor-middle becoming major-middle and the major-middle subsidiary in the ascending direction, or both becoming subsidiary in either or each direction, according as the number of new terms brought in in either or each direction shall be one or two or more than two.

In such a sorites the recognition of the truth of the premises, and of the necessity of the truth of the ultimate conclusion, is assumed as antecedent to further investigation in either direction.

For the purpose of illustration by geometrical plane figures, let it be assumed that eight new terms have been found, four (Y, Z, S, T,) successively in the ascending direction of the process of investigation, and four (K, Q, G, H,) successively in the descending direction.

The figures will now be as follows, the original figures 1 and 2 being again reproduced, and two additional quadrilaterals annexed to each, four of the points of which (two of each quadrilateral) in each case are designated by the symbols of the new terms as above.

Fig. 1 b

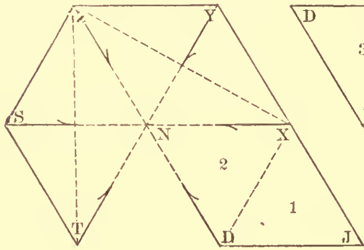
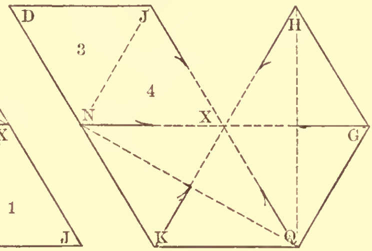


Fig. 2 b



The reasoning process in each direction will now most naturally (and, necessarily, geometrically considered and appropriately expressed) fall into the form of a compound epicheirema as follows.

Let $>$ signify "comprehends" and $<$ "is comprehended in."

In the descending direction as in Fig. 1b.

$T > S,$

$S > N; \because S > Z,$

and $Z > N; \because Z > Y,$

and $Y > N; \because Y > X,$

and $X > N; \because X > J,$

and $J > D,$

and $D > N;$

$\therefore T > N.$

Or thus :

$T > S,$

$S > Z,$

$Z > N; \because Z > Y,$

and $Y > X,$

and $X > N; \because X > J,$

and $J > D,$

and $D > N;$

$\therefore T > N.$

In the ascending direction as in Fig. 2b.

$H < G,$

$G < X; \because G < Q,$

and $Q < X; \because Q < K,$

and $K < X; \because K < N,$

and $N < X; \because N < D,$

and $D < J,$

and $J < X;$

$\therefore H < X.$

Or thus :

$H < G,$

$G < Q,$

$Q < X; \because Q < K,$

and $K < N,$

and $N < X; \because N < D,$

and $D < J,$

and $J < X;$

$\therefore H < X.$

But logically considered, the process may also be in the form of a compound sorites, in which the premises will be found in the order of the lines forming the perimeter of the figure in each case until the point is reached which would be the centre of the figure if it were drawn in the form of a hexagon, and the successive conclusions are alternate semi-diagonals pointing to the centre, the last being also the remaining line of the perimeter.

Such compound sorites fully expressed and in both directions are as follows :

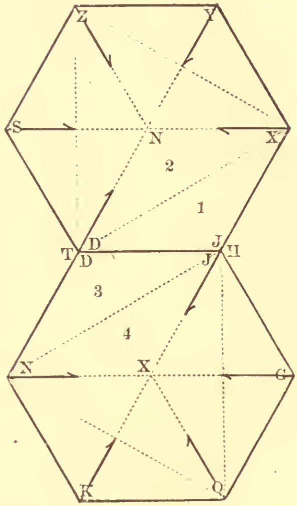
In the descending direction.	*In the ascending direction.*
T comprehends S,	H is comprehended in G,
S comprehends Z ;	G is comprehended in Q ;
— — — — — — — —	— — — — — — — — — —
Z comprehends Y,	Q is comprehended in K,
Y comprehends X ;	K is comprehended in N ;
— — — — — — — —	— — — — — — — — — —
X comprehends J,	N is comprehended in D,
J comprehends D,	D is comprehended in J,
D comprehends N ;	J is comprehended in X ;
∴ — — — — — — — —	∴ — — — — — — — — — —
X comprehends N ;	N is comprehended in X ;
∴ — — — — — — — —	∴ — — — — — — — — — —
Z comprehends N,	Q is comprehended in X,
and ∴ T comprehends N.	and ∴ H is comprehended in X.

The transverse diagonals in Figs. 1*b* and 2*b* are analogues of the unexpressed conclusions of the enthymemes of the third order into which the first four propositions of each compound sorites are divided by the first two dotted lines in each case, and which do not appear subsequently as expressed premises. The third dotted line in each compound sorites, with the character ∴ prefixed, signifies that the conclusion of the premises of the last included simple sorites immediately preceding is not expressed as such, but such conclusion follows immediately as the third premise—in connection with the two premises between the first and second dotted lines—of the first included simple sorites; of which also the conclusion as such is not expressed, but follows immediately as the third premise of the new principal simple sorites resulting from the whole process, composed of the first two and last two propositions in each case.

The ultimate conclusion of the foregoing compound sorites in the descending direction is "T comprehends N," and in the ascending direction is "H is comprehended in X." If now the two figures be considered as put together in like manner as Figs. 1 and 2 were put together to form Fig. 3 (page 16), namely, on their only common continuous line, J D descending, D J ascending, it will be found that the relation between the two extremes T and H, which can be logically demonstrated, cannot be geometrically established by means of the combined figures.

And if any two of the quadrilaterals, of which Figs. 3 and 19 are composed, are put together on the lines designated by the symbols of the middle terms of the sorites in each case with additional quadrilaterals annexed to each, it will be found that the combined figures will not serve as analogues of the compound sorites demonstrating the relation of the extremes reached by investigation in both directions.

Let now Figs. 1*b* and 2*b* be considered as redrawn—but with the original quadrilaterals in such form that each whole figure shall be exteriorly hexagonal—and put together on their common line as in the following figure :



The term of beginning of each original sorites is at an acute angle of the quadrilateral by which the sorites is represented instead of at an obtuse angle as in all previous figures, and the diagonal representing the unexpressed conclusion of the first which is the unexpressed premise of the second of the two syllogisms into which each sorites may be expanded is transverse relatively to that shown in previous figures, but which does not appear in these.

The figures considered separately or as combined are now in such form that they are perfect analogues of the compound processes of reasoning in so far as such processes may be exhibited on regular plane figures.

The premises of such process in the descending direction (on the combined figures) begin with T and follow the perimeter of the upper half of the figure until the point X is reached, and in the ascending direction begin with H and follow the perimeter of the lower half of the figure until the point N is reached, and then in each case follow the transverse diagonal in the original quadrilateral and continue thence along the whole perimeter of the other half of the figure until the ultimate term is reached ; and the successive conclusions are represented in the last half of the figure (first returning) by two transverse diagonals forming an angle, the vertex of which is at the point H in the descending direction and at the point T in the ascending—thence, *per saltum*, from the term of beginning of the original sorites in one direction to the term of beginning of the original sorites in the other, by means of the conclusion assumed to have been found in the original sorites in each case (but not expressed in the compound process) along a diagonal not shown in the figure connecting the terms of beginning of the two original sorites on the two original quadrilaterals—and are then further represented in the first half of the figure (last returning) by the line of its perimeter which does not represent a premise, then by a diagonal of the first half of the figure (of which diagonal only half is shown in the figure, but the arrow-head therein points to the ultimate term), and lastly by the line representing the original middle premise in both directions being the line common to each half of the figure.

The compound sorites thus described, fully expressed in both directions, are as follows:

In the descending direction.	*In the ascending direction.*
T comprehends S,	H is comprehended in G,
S comprehends Z;	G is comprehended in Q;
- - - - - - - -	- - - - - - - - - -
Z comprehends Y,	Q is comprehended in K,
Y comprehends X;	K is comprehended in N;
- - - - - - - -	- - - - - - - - - -
X comprehends D,	N is comprehended in J,
D comprehends N;	J is comprehended in X;
- - - - - - - -	- - - - - - - - - -
N comprehends K,	X is comprehended in Y,
K comprehends Q;	Y is comprehended in Z;
- - - - - - - -	- - - - - - - - - -
Q comprehends G,	Z is comprehended in S,
G comprehends H;	S is comprehended in T;
∴ - - - - - - - -	∴ - - - - - - - - - -
Q comprehends H;	Z is comprehended in T;
∴ - - - - - - - -	∴ - - - - - - - - - -
N comprehends H;	X is comprehended in T;
∴ - - - - - - - -	∴ - - - - - - - - - -
X comprehends H;	N is comprehended in T;
∴ - - - - - - - -	∴ - - - - - - - - - -
Z comprehends H,	Q is comprehended in T,
and ∴ T comprehends H.	and ∴ H is comprehended in T.

The principal simple sorites resulting from each of the foregoing compound sorites are not correlatives of each other, although reaching the same ultimate conclusion, or rather conclusions which are convertible into each other. But the typical simple sorites, hereinbefore described, as represented in the figure (not in the sorites) would be as follows (progressive and regressive in each direction), and are correlatives of each other:

Progressive descending.	*Regressive ascending.*
T comprehends J,	J is comprehended in T,
J comprehends D,	D is comprehended in J,
D comprehends H;	H is comprehended in D;
∴ T comprehends H.	∴ H is comprehended in T.

Progressive ascending.	*Regressive descending.*
H is comprehended in D,	D comprehends H,
D is comprehended in J,	J comprehends D,
J is comprehended in T;	T comprehends J;
∴ H is comprehended in T.	∴ T comprehends H.

The typical simple sorites is represented by one and the same line in the figure, the first premise in each (progressive in each direction) being represented by such line, considered as designated by a symbol without and a symbol within the figure, the second and middle premise by the two symbols within the figure, the third premise by a symbol within and a symbol without the figure, and the conclusion by the two symbols without the figure.

But the typical simple sorites as represented in the compound sorites (not in the figure) would be as follows:

Progressive descending	*Regressive ascending.*
T comprehends X,	X is comprehended in T,
X comprehends N,	N is comprehended in X,
N comprehends H;	H is comprehended in N;
∴ T comprehends H.	∴ H is comprehended in T.

Progressive ascending.	*Regressive descending.*
H is comprehended in N,	N comprehends H,
N is comprehended in X,	X comprehends N,
X is comprehended in T;	T comprehends X;
∴ H is comprehended in T.	∴ T comprehends H.

It will now be remembered that the original processes in both directions were assumed as having been gone through with, and their respective ultimate conclusions established antecedently to further investigation in either direction. The terms of such conclusions may therefore be considered as representing a concrete genus and a concrete species proximate to each other, or so nearly so that the relation of each to the other is recognized and admitted. Thus the two forms of the typical simple sorites represented by, and taken from, both the figure and the compound sorites, are justified.

Let now the parallelogram represented in the figure by triangles 1 and 3 be considered as taken out, and let the two remaining parts of the figure be considered as put together on the lines of the two transverse diagonals in the original quadrilaterals as common to both.

The two premises in each of the compound sorites, namely, "X comprehends D" and "D comprehends N" in the descending direction, and "N is comprehended in J" and "J is comprehended in X" in the ascending direction, will now be supplanted by their two conclusions respectively which would read thus, "But X comprehends N" and "But N is comprehended in X," and let the word "and" be prefixed to the next proposition in each case. The typical simple sorites above given, as taken from the two compound sorites as thus changed in form, are thereby further justified. But the new figure and the compound sorites will both be irregular in form, the latter consisting of the premises of one process in each case conjoined — by the proposition thus substituted—to the premises of another process originally in the opposite direction, but the direction changed so that the whole process with the successive conclusions following may be in one and the same direction.

The typical regressive simple sorites in both forms may be reduced to the form of simple syllogisms as follows:

As taken from the figure,

In the descending direction.	*In the ascending direction.*
D comprehends H,	J is comprehended in T,
T comprehends D;	H is comprehended in J;
∴ T comprehends H.	∴ H is comprehended in T.

As taken from the compound sorites,

In the descending direction.	*In the ascending direction.*
X comprehends H,	X is comprehended in T,
T comprehends X;	H is comprehended in X;
∴ T comprehends H.	∴ H is comprehended in T.

The premises of these syllogisms in each case are not correlatives of each other, although the conclusions are, and it is only by means of the simple sorites, in the middle premise of which the two middle terms of such syllogisms in each case are compared, that the complete correlation of the processes of reasoning in both directions can be exhibited. The claim of the simple sorites, singly or two combined, to be regarded as the complete and (considered as combined) necessary form of the process of reasoning is thus vindicated.

Thus far only can the analogy between compound logical processes and combinations of regular geometrical plane figures be exhibited on paper, but the process in respect to each direction of the process of investigation may be considered as continued indefinitely in a circular direction about the central points in Figs. 1*b* and 2*b*, either as originally drawn or as redrawn; but if investigation shall have been made in both directions, then such investigation and the reasoning processes in respect thereto, in so far as they may be represented by a combination of regular geometrical figures, are limited as shown in the foregoing figure.

But on combined irregular figures, as before described, with other like figures annexed laterally in both directions (proceeding upwardly to the right and downwardly to the left), the compound processes may be logically pursued indefinitely.

If the original quadrilaterals in Figs. 1*b* and 2*b* had been in the form of squares, as in Figs. 4 and 5 on page 17, the annexed quadrilaterals would have been also in like form, but there would have been three in each figure instead of two as in Figs. 1*b* and 2*b*, and the figures would have been perfect analogues of compound processes of reasoning in each direction, in like manner as Figs. 1*b* and 2*b*, with two additional terms brought in in each direction.

But if the two figures are considered as put together on the lines J D and D J, it will be found that the resulting figure is not a perfect analogue of the compound reasoning processes establishing the relation of the two extremes reached by investigation in both directions. The conclusions of two of the included sorites would not be represented in the first half of the figure (last returning) in each direction by lines of the figure, but would be reached by indirection. Thus the square in respect to such combined figures is an imperfect analogue of the sorites.

The accompanying figures, 3*b* and 19*b* (on a folded sheet following page 134), exhibit the original combinations of quadrilaterals, as in Figs. 3 (page 16) and 19 (page 35), considered as the faces of tetrahedra of edge 1 spread out, with five additional combinations annexed to each, five new terms being assumed to have been found by investigation in each direction, two in each of the first two annexed combinations and one in the third, the latter being found in the third annexed combination to be related to the fourth term of the sorites on the faces of the original combination, but not at the same point, both the original complete processes of reasoning being assumed to have been gone through with and their respective ultimate conclusions established, antecedently to further investigation in either direction. The course of the process of investigation will now be found by an examination of the figures to have been ascending from X, in Fig. 3*b*, along the lines which are analogues of the premises of the syllogisms represented by the several triangles marked 3; and descending from N′, in Fig. 19*b*, along the similar lines of the triangles marked *c*, until in the first case the ultimate highest point X′ was reached, and in the second case the ultimate lowest point N.

The compound processes of reasoning, retracing these lines respectively, descend in the first case from X′ to X, and thence, along the similar lines of triangles 1 and 3 in the original combination, to the ultimate point N, and in the second case ascend in like manner from N to N′, and thence, along the similar lines of triangles *a* and *c* in the original combination, to the ultimate point X′. Such processes constitute the following compound sorites fully expressed in both directions.

In the descending direction as in Fig. 3b.

X′ comprehends D′,
D′ comprehends X′;

```
– – – – – – – –
```

N′ comprehends J,
J comprehends **N**:

```
– – – – – – – –
```

N comprehends P,
P comprehends T;

```
– – – – – – – –
```

T comprehends S,
S comprehends Z;

```
– – – – – – – –
```

Z comprehends Y,
Y comprehends X;

```
– – – – – – – –
```

X comprehends J,
J comprehends D,
D comprehends N;

```
∴ – – – – – – – –
```

X comprehends N;

```
∴ – – – – – – – –
```

Z comprehends N;

```
∴ – – – – – – – –
```

T comprehends N;

```
∴ – – – – – – – –
```

N comprehends N;

```
∴ – – – – – – – –
```

N′ comprehends N,
and ∴ X′ comprehends N.

In the ascending direction as in Fig. 19b.

N is comprehended in J,
J is comprehended in X;

```
– – – – – – – – – –
```

X is comprehended in D′,
D′ is comprehended in **X′**;

```
– – – – – – – – – –
```

X′ is comprehended in B,
B is comprehended in H;

```
– – – – – – – – – –
```

H is comprehended in G,
G is comprehended in Q;

```
– – – – – – – – – –
```

Q is comprehended in K,
K is comprehended in N′;

```
– – – – – – – – – –
```

N′ is comprehended in D′,
D′ is comprehended in J′,
J′ is comprehended in X′;

```
∴ – – – – – – – – – –
```

N′ is comprehended in X′;

```
∴ – – – – – – – – – –
```

Q is comprehended in X′;

```
∴ – – – – – – – – – –
```

H is comprehended in X′;

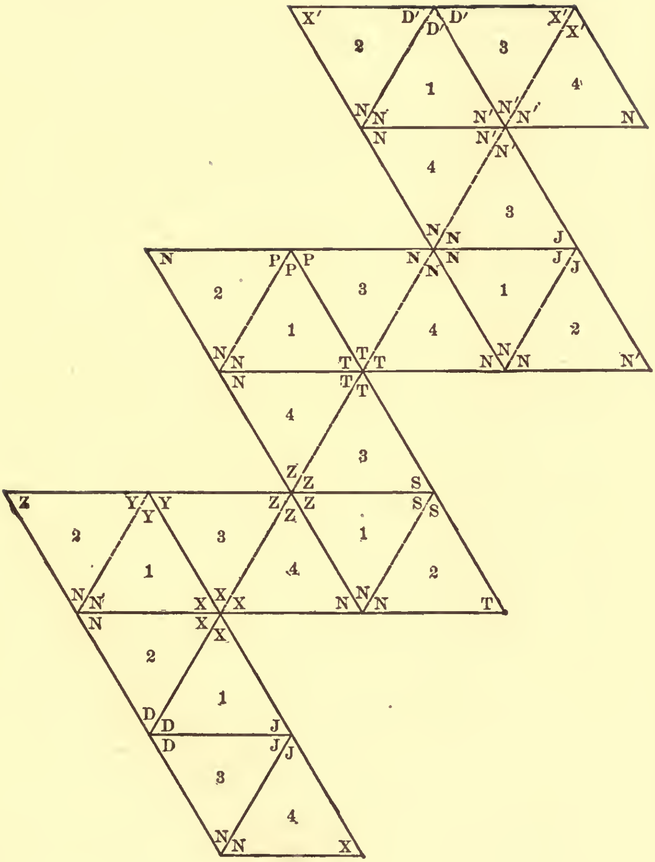
```
∴ – – – – – – – – – –
```

X′ is comprehended in X′;

```
∴ – – – – – – – – – –
```

X is comprehended in X′,
and ∴ N is comprehended in X′.

Fig. 3 b



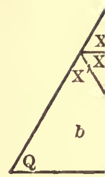
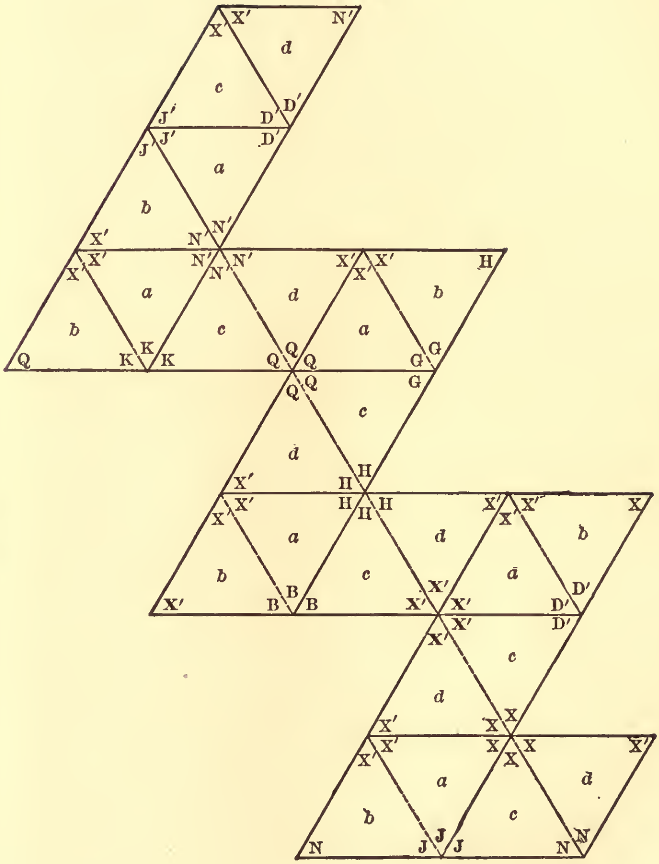


Fig. 19 b


X'
N'
X
d
c
J'
D'
D'
J'
D'
J'
a
b
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Two of the unexpressed conclusions, but expressed as premises (one in each compound sorites), namely, "**N** comprehends N" in the descending direction, and "**X′** is comprehended in X′" in the ascending direction, have the symbols designating their terms alike in each case, but by reference to the figures such symbols will be found to designate two different points in each case, and therefore two different terms. These symbols, where they designate exterior points of the octastron respectively, are put in full-faced type in the figures and also in the sorites.

But the process of reasoning may be exhibited in a shorter form in each case, than by retracing the lines of the process of investigation, namely, as follows:

In the descending direction.	*In the ascending direction.*
X′ comprehends N′,	N is comprehended in X,
N′ comprehends **N**,	X is comprehended in **X′**,
N comprehends T,	**X′** is comprehended in H,
T comprehends Z,	H is comprehended in Q,
Z comprehends X;	Q is comprehended in N′;
∴ X′ comprehends X.	∴ N is comprehended in N′.
– – – – – – – –	– – – – – – – – – –
But X comprehends N,	But N′ is comprehended in X′,
and ∴ X′ comprehends N.	and ∴ N is comprehended in X′.

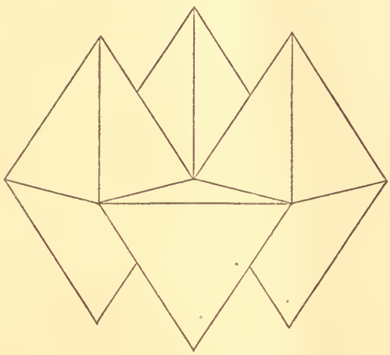
Each process is in the form of a compound sorites, not fully expressed, to which is appended an enthymeme consisting of the ultimate conclusion of the original principal sorites as a premise, and the conclusion resulting therefrom and the preceding conclusion of the compound sorites considered as the suppressed premise of the enthymeme.

The analogues of the premises of the compound sorites in each case will be found to consist of the dotted diagonals, by which one of the quadrilaterals of each annexed combination is divided into triangles 3 and 4, in Fig. 3*b*, and *c* and *d*, in Fig. 19*b*, and which are also analogues of the conclusions of the processes on triangles 3 and *c* respectively considered as conducted *pari passu* with the process of investigation, and the ultimate conclusions of the compound sorites, "X′ comprehends X" descending, and "N is comprehended in N′" ascending, are represented in the figures by all such dotted diagonals in each figure respectively, forming in each case one and the same straight line connecting the ultimate point reached with the point of beginning of investigation.

The process of investigation in Fig. 3*b* is ascending, and in Fig. 19*b* descending, and in strict accordance with the forms of logic the processes of reasoning should have been in the regressive configuration in the same directions respectively (into which the processes as shown are convertible—see page 16), the process of investigation being always progressive, and that of reasoning in retracing the steps regressive, but in such case the compound process, when it should reach the principal combination, would require that the direction of the original process, assumed to have been gone through with thereon, should be changed, or otherwise there would be in each case a combination of processes in opposite directions.

If now the figures be drawn upon and cut from cardboard, and the board cut half-way through on all the lines (on the interior lines of each combination, on the face of the figure as shown, but on the lines connecting the combinations, on the other side), each of the figures may be folded so as to inclose six volumes of space, each in the form of a regular tetrahedron, connected each two by an edge in such manner that the line of each connecting edge shall be perpendicular (on both sides) to a plane in which the opposite edges of the two connected tetrahedra shall lie.

The figures cut and folded as described will be found to take the form shown in the following illustration:



And let the edges of the first and last tetrahedra, which will come together, be fastened together.

Such edges in the descending direction (Fig. 3*b*) are D N, common to triangles 2 and 3 of the original combination of quadrilaterals (when folded), and X′ N, common to triangles 2 and 4 of the last annexed combination; and in the ascending direction (Fig. 19*b*) are J′ X′, common to triangles *b* and *c* of the original combination, and N X′, common to triangles *b* and *d* of the last annexed combination. The outer point, common to the two edges brought together in the folding of each figure, is designated by two symbols, viz.: D and X′ in the descending direction and J′ and N in the ascending.

In like manner, as shown in respect to the quadrilaterals considered as plane figures, the combined and folded quadrilaterals have each made a complete circuit, and the analogy between the logical processes and geometrical solid figures cannot be further exhibited, but the processes may be considered as further indefinitely continued proceeding successively along the lines of the surfaces of the same volumes of space. But in plane figures exhibiting the faces of the solid figures spread out they may be represented on paper indefinitely.

The circuit thus formed in each case is similar, in respect to the relative positions to each other of the tetrahedra of which it consists, to six of the eight tetrahedra of which with six octahedra an octahedron of edge 2 is composed, in which considered as contained within a right triangular pyramid of edge 4, the centre of the pyramid may be reached as described on pages 62 and 63.

The exterior face of the original tetrahedron in circuit 3*b* is triangle 1, and in circuit 19*b* is triangle *a*, and that of each of the annexed tetrahedra in the former is triangle 3 and in the latter triangle *c*. Thus the courses of the processes of investigation and reasoning along the lines of such faces until the original tetrahedron is reached returning (on the faces of which the processes were assumed as gone through with antecedently to further investigation) are wholly on the surface, the lines of the shorter processes being the boundaries in each case of the hexagon composed of the interior faces of the tetrahedra of the circuit opposite their exterior vertices respectively.

Let now the eight superposed tetrahedra of the oct·-astron be considered each as named by the symbol at its exterior vertex, as in Figs. 17 and 18 (page 34), and let a card-board figure of the included octahedron be considered as in hand together with the two circuits of card-board tetrahedra formed by the folding of the foregoing figures.

If now the two circuits of tetrahedra be applied to the octahedron in such manner that the first and principal tetrahedron of each formed by the folding of the original combination of quadrilaterals shall occupy the positions respectively of tetrahedra X and N′ relatively to each other, as in the oct·astron held as shown in Fig. 17, they will be found, circuit 3*b* descending backwardly, but obliquely, to the right, and circuit 19*b* ascending forwardly and obliquely to the left, the second, third, and fourth tetrahedra in each circuit being entirely outside of the oct·astron, and the fifth and sixth in circuit 3*b* being N and D′, and in circuit 19*b* X′ and J. Six faces of the octahedron will have been covered, leaving only those exposed on which in the completed oct·astron tetrahedra D and J′ are superposed.

The principal simple sorites resulting from the two fully expressed compound sorites, on page 134, are as follows:

In the descending direction.	*In the ascending direction.*
X′ comprehends D′,	N is comprehended in J,
D′ comprehends N′,	J is comprehended in X,
N′ comprehends N;	X is comprehended in X′;
∴ X′ comprehends N.	∴ N is comprehended in X′.

And the principal simple syllogisms resulting from the shorter processes on page 135, in which the successive conclusions of the reasoning process conducted *pari passu* with the process of investigation are employed as premises, are as follows :

In the descending direction.	*In the ascending direction.*
X′ comprehends X,	N is comprehended in N′,
X comprehends N;	N′ is comprehended in X′;
∴ X′ comprehends N.	∴ N is comprehended in X′.

Thus, from two diametrically opposite stand-points, as shown by the two circuits applied to the octrahedron, and by paths entirely diverse, the same ultimate result is reached, but expressed, in the one case descending from above, as the greater comprehending the less, and in the other case, ascending from beneath, as the less comprehended in the greater.

If now the two complete processes be considered each as reversed in direction, so that the first, instead of descending from X at the upper pole of the axis of revolution to N on the lower horizontal line (see Fig. 18, page 34), shall ascend from N to X, and the second, instead of ascending from N′ at the lower pole of such axis to X′ on the upper horizontal line (see Fig. 17, page 34), shall descend from X′ to N′, and if figures should be drawn showing the processes so conducted, such figures would be appropriately numbered Figs. 3*c* and 19*c*, and would be similar to Figs. 3*b* and 19*b* except that the original combination would be the upper combination in the former and the lower in the latter, and the annexed combinations in each case would proceed to the left instead of to the right. The faces of the principal and first annexed combinations which would adjoin each other would be 4 and 3 in Fig. 3*c*, instead of 2 and 1 as in Fig. 3*b*, and *d* and *c* in Fig. 19*c*, instead of *b* and *a* as in Fig. 19*b*. The symbols, except the last, employed in the first three annexed combinations in Fig. 3*c*, may be those of Fig. 19*b*, and in Fig. 19*c* those of Fig. 3*b*, but marked as primes in each case, their employment as such serving to show their significations as comprehending or comprehended as they were originally employed. Fig. 3*c* thus becomes an ascending circuit and Fig. 19*c* descending.

Such figures could be folded in the form of circuits of tetrahedra and applied to the faces of an octahedron similarly to Figs. 3*b* and 19*b*, but the tetrahedra formed by the folding of the original combinations would occupy the positions of tetrahedra N and X′ of the oct·astron. By such application the faces of the octahedron left exposed, as described on page 139, would be covered and instead there would be left exposed the faces on which in the completed oct·astron tetrahedra J and D′ are superposed. The exterior faces of the circuits, instead of being 1 and 3 as in Fig. 3*b*, and *a* and *c* as in Fig. 19*b*, as described on page 138, would be 3 and 1 in Fig. 3*c* and *c* and *a* in Fig. 19*c*.

The principal simple sorites which would result from the compound sorites in such case would be as follows:

In the ascending circuit as would be shown in Fig. 3c.	*In the descending circuit as would be shown in Fig. 19c.*
N′ is comprehended in J′,	X comprehends D,
J′ is comprehended in X′,	D comprehends N,
X′ is comprehended in X;	N comprehends N′;
∴ N′ is comprehended in X.	∴ X comprehends N′.

And the principal simple syllogisms which would result from the shorter compound processes would be as follows:

In the ascending circuit.	*In the descending circuit.*
N′ is comprehended in N,	X comprehends X′,
N is comprehended in X;	X′ comprehends N′:
∴ N′ is comprehended in X.	∴ X comprehends N′.

Thus the relations to each other of like extremes of each of the two complete processes of reasoning on the faces of the oct·astron are established, namely, of N and X′ (the original ultimate terms) to each other as in the processes previously shown, and of X and N′ (the original terms of beginning, but ultimate in the reverse directions) to each other as in the processes of which the results in simple form have been just shown, and by examining the principal simple sorites resulting from the compound sorites in each case it will be seen that the combination of the two complete processes of reasoning in one harmonious whole is accomplished in each direction through the third term in conjunction with the fourth in the same direction, being the second and first in the opposite direction, as middle terms, and all other terms brought in in the full processes are subsidiary.

But the two complete processes of reasoning of the like extremes of which the relations to each other are thus established are not those on the faces of the oct·astron considered as consisting of two intervolved tetrahedra, but as consisting of eight tetrahedra superposed on the faces of an octahedron and having their points (that is, all the points of each) designated by the logical symbols similarly to those of the intervolved tetrahedra, and so superposed relatively to each other that the whole figure shall have its exterior points designated similarly to the oct·astron considered as consisting of two intervolved tetrahedra, as described in § 11 on page 37.

By examining the tetrahedra of the circuits 3*b* and 19*b*, it will be found that the new principal simple sorites resulting from the two compound sorites are the simple sorites represented on the faces of the last annexed tetrahedron in each case, and in like manner it would be found in respect to circuits 3*c* and 19*c* if those figures were drawn and folded as hereinbefore described, and the compound sorites represented thereon fully expressed.

Mathematically considered, each link of the chain of reasoning is represented by a triangle and is limited to three terms, but logically considered each link is represented by a combination of quadrilaterals on which the reasoning is exhibited as both descending and ascending (but expressed only in either direction), and extends, but is limited to four terms.

Let now the superposed tetrahedra be considered each as having its exterior point designated by the symbol which is the term of beginning of the combined sorites of which it is the analogue, namely, the four which together, with the included octahedron, constitute the intervolved tetrahedron in the descending direction, each by the symbol X, and the four in the ascending direction each by N′. It will now be necessary to add a separate appellation to each of such terms other than the two originals, so as to distinguish them from the originals respectively and from each other, which appellations it will be necessary also to add to each of the other terms.

Let the following be the full symbols of the terms in their regular order in each case, the column at the left hand showing the designation of the first term on the faces of the intervolved tetrahedra of the oct·astron :

Of the tetrahedra considered as descending.

X	X J D N
J	X_2 J_2 D_2 N_2
D	X_4 J_4 D_4 N_4
N	X_6 J_6 D_6 N_6

Of the tetrahedra considered as ascending.

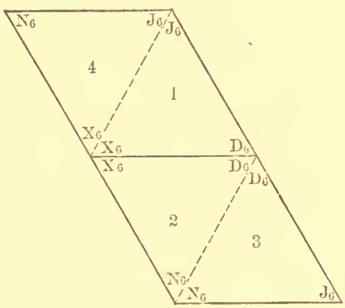
N′	N′ D′ J′ X′
D′	N_3 D_3 J_3 X_3
J′	N_5 D_5 J_5 X_5
X′	N_7 D_7 J_7 X_7

If now figures were drawn corresponding to Figs. 3*b* and 19*b* (and which would be appropriately numbered Figs. 3*d* and 19*d*), it is manifest that the processes of investigation and reasoning along the lines of the first three consecutively annexed combinations of quadrilaterals in such figures and the designations of the points of such quadrilaterals (except of one point in the third annexed combination in each case, which will have to be changed, as will be hereinafter shown) would be the same as in Figs. 3*b* and 19*b*.

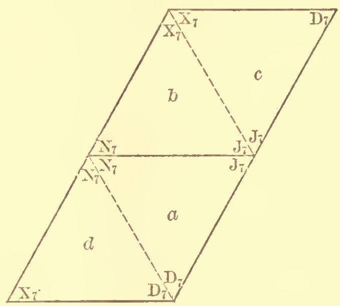
But when the fourth annexed combination is reached in each case it will be found that although the physical construction of the circuit may be the same, and spread out in plane figures as in Figs. 3*b* and 19*b*, it will be necessary to change the order of the faces as to their numbered or lettered designations, so that the exterior points of the tetrahedra formed by the folding of such annexed combination in each case shall be X_6 instead of N, and N_7 instead of **X′**, as in the oct·astron considered as consisting of intervolved tetrahedra, and as shown in the figures, and accordingly the face of the fourth annexed combination adjacent to face 4 of the third must be face 2, in Fig. 3*d*, instead of face 1 as in Fig. 3*b*, and the face adjacent to face *d* must be face *b* in Fig. 19*d*, instead of face *a* as in Fig. 19*b*. These changes will now require the substitution of X_6 in place of N, in the third annexed combination in Fig. 3*d*, and of N_7 in place of X′, in the like combination in Fig. 19*d*, and the points of the fourth annexed combination in Fig. 3*d* will be designated X_6, J_6, D_6, and N_6, and in Fig. 19*d* N_7, D_7, J_7, and X_7.

The fourth annexed combination in the two figures will now be as follows:

In Fig. 3d.



In Fig. 19d.



These figures being considered as substituted in Figs. 3*b* and 19*b*, and the symbol designating one of the points in the third annexed combination of quadrilaterals in each case changed as above described, and all the symbols in the last combination obliterated, the figures may be folded as before and the geometrical processes may be pursued if the symbols be regarded as having no signification other than as designating the points, but if they are considered as retaining their logical significations, then all reasoning is at an end when the fourth annexed combination is reached.

It will be necessary only to consider the syllogisms on face 2 in Fig. 3*d* and *b* in Fig. 19*d*. They may be considered in two ways; first, in the order in which they have been hitherto considered as proceeding (but on face 1 in 3*b* and *a* in 19*b*), in which case one of the premises will be found to be untrue although the conclusion will be true, and, secondly, in the order in which the premises are both true, but the conclusion (all the propositions being, logically considered, universal-affirmative), although still true, will be found to be unwarranted.

Thus, in the order in which they have been hitherto considered as proceeding:

On face 2, in Fig. 3d.	*On face b, in Fig. 19d.*
X_6 comprehends N_6,	N_7 is comprehended in X_7,
D_6 comprehends X_6;	J_7 is comprehended in N_7;
∴ D_6 comprehends N_6.	∴ J_7 is comprehended in X_7.

Here the second premise in each syllogism is untrue, although the conclusion in each is true.

And, secondly, in the order in which the premises are both true:

On face 2, in Fig. 3d.	*On face b, in Fig. 19d.*
X_6 comprehends N_6,	N_7 is comprehended in X_7,
X_6 comprehends D_6;	N_7 is comprehended in J_7;
∴ D_6 comprehends N_6.	∴ J_7 is comprehended in X_7.

Here the syllogism in each case is in the third figure of logic in which only a particular conclusion can be deduced, which is not the case in either of the syllogisms as stated. The conclusions, therefore, although still true, are unwarranted.

Thus it will be seen that the two complete processes of reasoning on the faces of an oct·astron, considered as consisting of tetrahedra superposed upon an octahedron, cannot be linked together when the exterior points of the oct·astron are designated by the terms of beginning of the processes respectively, but only when designated similarly to the points of the oct·astron considered as consisting of two intervolved tetrahedra; and, on the other hand, the two processes on the faces of the intervolved tetrahedra cannot be linked together so that the relations of their extremes designating opposite poles of each of two axes of the oct·astron shall be established. But they can be so linked together on the surface of the circumscribed sphere of the oct·astron considered as consisting of both intervolved and superposed tetrahedra, as will be herein next shown. So wonderfully and perfectly is nature consistent with herself throughout her whole domain.

The sphere as described in § 18 (on page 109) *et seq.*, is the circumscribed sphere of the oct·astron considered as consisting of two intervolved tetrahedra.

Let it now be considered as the circumscribed sphere of the oct·astron consisting also of tetrahedra superposed upon the included octahedron, as described in § 11, on page 37.

The exterior points of the oct·astron so considered will be designated on the surface of the sphere in like manner as before—see Figs. 34 and 35, on pages 111 and 112—but those points of the tetrahedra considered as superposed which fall upon the included octahedron do not come to the surface of the sphere, and their designations will not, therefore, appear, unless brought to the surface through some of the figures thereon, which, it will be observed, are not figures of the oct·astron, but of curved sections of the added figures in the construction of the octahedron of the second order. They may accordingly and appropriately be considered as brought to the surface through the rhombs at their obtuse angles, and Figs. 34 and 35 are herewith reproduced with such obtuse angles so designated, except that Fig. 35 is exhibited as held above the eye, as described in the text following that figure.

Both the figures are considered as representing the sphere held with its axis X N′ vertical in such position below the eye in Fig. 34*a* that the line of vision shall be perpendicular to the surface at the point X′ and shall coincide with the axis X′ N, and above the eye in Fig. 35*a* that such line shall be perpendicular to the surface at the point N and shall coincide with the same axis N X′.

The syllogism on face 1 of each of the three tetrahedra named X, J, and D, and the three syllogisms on faces *c*, *b*, and *d* of the tetrahedron named X′ are brought to the surface in Fig. 34*a*, and the syllogism on face *a* of each of the three tetrahedra named N′, D′, and J′, and the three syllogisms on faces 3, 2, and 4 of the tetrahedron named N are brought to the surface in Fig. 35*a*.

It will be necessary only to consider such syllogisms on the upper rhomb in Fig. 34*a* and the lower one in Fig. 35*a*. They are considered, those in Fig. 34*a* both as descend-

Fig. 34a

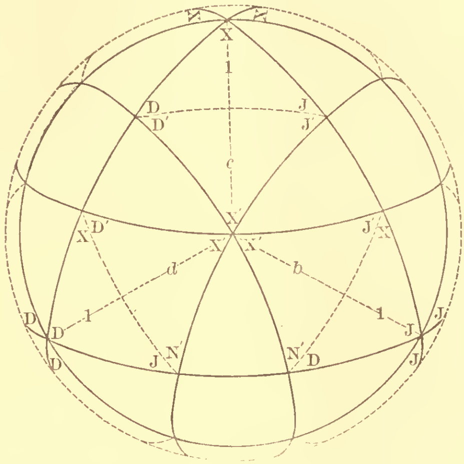
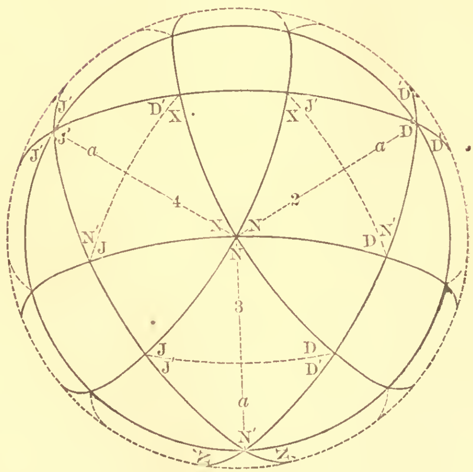


Fig. 35a



ing and those in Fig. 35*a* both as ascending, accordingly as they are viewed from above or below the sphere.

They are as follows:

In Fig. 34a.

X comprehends J,	X′ comprehends J′,
J comprehends D,	J′ comprehends D′,
∴ X comprehends D.	∴ X′ comprehends D′.

In Fig. 35a.

N′ is comprehended in D′,	N is comprehended in D,
D′ is comprehended in J′,	D is comprehended in J,
∴ N′ is comprehended in J′.	∴ N is comprehended in J.

The conclusion of each of these syllogisms establishes the relation between the first and third terms of the combined sorites descending or ascending throughout on the faces of each superposed tetrahedron considered as a whole figure.

Let now such conclusion in each case be taken as the first premise of the syllogism represented by one of the triangles formed by the transverse diagonal in each case in like manner as in the figure on page 126 representing the sorites as exhibited on plane geometrical figures, and there will be found the following four syllogisms:

In Fig. 34a.

X comprehends D,	X′ comprehends D′,
D comprehends X′,	D′ comprehends X,
∴ X comprehends X′.	∴ X′ comprehends X.

In Fig. 35a.

N′ is comprehended in J′,	N is comprehended in J,
J′ is comprehended in N,	J is comprehended in N′,
∴ N′ is comprehended in N.	∴ N is comprehended in N′.

Thus the equality to each other of the maximus terms of both processes is established in Fig. 34*a*, and in like manner that of the magnus terms in Fig. 35*a*.

Let now the foregoing syllogisms considered as the ultimate results of the processes on the faces of the superposed tetrahedra be combined with the conclusions of the processes on the faces of the intervolved tetrahedra and the resulting conclusions will be found to establish the relations of the terms by which opposite poles of each of two axes of the oct astron, and of its circumscribed sphere are designated.

Such combinations of processes will be as follows:

In Fig. 34a.

	X	comprehends	D,		X′	comprehends	D′,
	D	comprehends	X′,		D′	comprehends	X,
∴	X	comprehends	X′.	∴	X′	comprehends	X.
	— — —	— — —	— — —		— — —	— — —	— — —
But	X′	comprehends	N′,	But	X	comprehends	N,
and ∴	X	comprehends	N′.	and ∴	X′	comprehends	N.

In Fig. 35a.

	N′	is comprehended in	J′,		N	is comprehended in	J,
	J′	is comprehended in	N,		J	is comprehended in	N′,
∴	N′	is comprehended in	N.	∴	N	is comprehended in	N′.
	— — —	— — —	— — —		— — —	— — —	— — —
But	N	is comprehended in	X,	But	N′	is comprehended in	X′,
and ∴	N′	is comprehended in	X.	and ∴	N	is comprehended in	X′.

All the foregoing syllogisms, as indeed also all the syllogisms previously exhibited except those which are regressive, are in the fourth figure of logic and it may be objected that the conclusions are therefore unwarranted.

But all the terms, whether employed as subject or predicate, are herein considered as distributed, and the fourth figure in such case is the natural and therefore perfect figure of logic. But the premises, in order to make the syllogisms conform to the first figure of logic, may be considered as transposed, in which case all the syllogisms will be regressive.

To return now to the consideration of the circuits. They have been considered as applied to the octahedron in pairs, 3*b* and 19*b* together and 3*c* and 19*c* together, in each of which cases each circuit is wholly independent of the other; and in the case of each circuit there are three tetrahedra entirely without the oct·astron considered as complete.

Let them now be considered as applied to the octahedron in pairs, as follows: 3*b* and 3*c* together and 19*b* and 19*c* together.

It will now be found that the two circuits in each case are not independent of each other, but are interdependent, four of the tetrahedra of each being common to both, namely, two of the tetrahedra which are tetrahedra of the oct·astron—in the case of 3*b* and 3*c* X and N (so named) descending and N and X ascending, and in the case of 19*b* and 19*c* N′ and X′ ascending and X′ and N′ descending—and two of the tetrahedra in each case which are without the oct·astron, namely, those which are connected by their edges with the above-named tetrahedra being the first and third of the outside tetrahedra (but second and fourth of the circuit) in each direction. The third of the tetrahedra of the circuits which are also tetrahedra of the oct·astron (sixth of the circuits) are, as has been before seen in the case of 3*b*, the tetrahedron named D′ and in the case of 19*b*, J, and upon examination of 3*c* would be found to be J′, and in the case of 19*c*, D; and the second of each of the outside tetrahedra in each circuit (third of the circuit) is different from the second in the other.

Thus, by the application of the four circuits to the octahedron, all the faces of the latter have been covered, and the two intervolved tetrahedra of the oct·astron are complete with four outside tetrahedra annexed to each, making eight on the two conjoined and considered as descending from X and ascending from N′, the poles of one axis, and as ascending from N and descending from X′, the poles of another axis.

The combinations of the interdependent circuits, 3*b* and 3*c* together, and 19*b* and 19*c* together, consist each of the eight tetrahedra, of which with six octahedra an octahedron of edge 2 is composed, as described on pages 62 and 63 and referred to on page 138.

The centre of each combination is common to the two circuits of the combination and is designated by the symbols of the ultimate terms of the two processes on the faces of the two principal superposed tetrahedra, the first of each circuit.

The eight outside tetrahedra will be found by a careful examination of the circuits to be connected by their edges (one of each) with the eight tetrahedra of the oct·astron by one of their exterior edges respectively, namely, those in circuits 3*b* and 3*c* with tetrahedra X, J′, N, and D′ and those in circuits 19*b* and 19*c* with tetrahedra N′, D, X′, and J. The edges of the intervolved tetrahedra which are thus connected with the outside tetrahedra in the two interdependent circuits 3*b* and 3*c* are X N and J′ D′, and in the two interdependent circuits 19*b* and 19*c* N′ X′ and D J. If the oct·astron be considered as the nucleus of a cube, such edges in pairs as above will be found to be the diagonals of two opposite faces of the cube.

But there are three exterior edges of each of the superposed tetrahedra of the oct·astron, and it will now be manifest that if the oct·astron be approached on either one of its other two sides, faces 2 and 4 descending, or *b* and *d* ascending, the side on which it is approached may have the points of its face designated by the symbols of triangle 1 or *a* (as the case may be) and the symbols of the other triangles (in both directions) will take their appropriate places on the other faces accordingly. The symbols of the terms of beginning of the two complete processes will be at the same points, but considered as the stand-points from which the processes are to be conducted, on different faces in each case from those in which they have been hitherto considered, but the symbols of all the other terms will not only be on different faces, but also at different points. The processes being now considered as conducted from such stand-points successively, the outside tetrahedra of the new circuits which would be formed would be connected with the tetrahedra of the oct·astron by edges of the latter respectively different in each case from those before described, beginning with the edge opposite the face on which the reasoning is considered as beginning, thus bringing the number of such outside tetrahedra up to twenty-four.

The whole figure formed by all the circuits consists of the oct·astron and the twenty-four tetrahedra mentioned in the description of the construction of an octahedron of the second order on page 51, and may be called the skeleton of such octahedron.

By the application of all the circuits to the octahedron their centres are found, N in 3*b* and X in 3*c* at the middle point of edge X N of the intervolved tetrahedron in the descending direction, and X′ in 19*b* and N′ in 19*c* at the middle point of edge N′ X′ of the intervolved tetrahedron in the ascending direction, such middle points being opposite poles—in each case of processes conducted from different stand-points—of one of the three axes of the octahedron. Thus it will be seen that the points of the octahedron can only be considered as designated by the symbols of the ultimate extremes of the two processes.

But two of the points only, in each case, and not the centre, of the octahedron have been reached.

Comparing the cube as it was considered with reference to its construction on pages 10 and 44, with the oct·astron and the circuits of tetrahedra, it will be observed that it differs from each of them in this important respect, namely, that in the oct·astron the faces of the tetrahedra are superposed upon an octahedron and in the circuits may be considered as so superposed, the volumes of space left between the tetrahedra of the circuits and all outer space, in all cases inviting the interposition therein of other octahedra (all the circuits being considered as having been gone through with), with the evident design on the part of nature that their outer faces can be in like manner proceeded along and built upon, but in the cube (edge 1) the faces of its included tetrahedron or (edge 2) of the tetrahedra of its included oct·astron have superposed thereon other and irregular tetrahedra, there being, in the latter case, but one octahedron, the points of which, only, come to the surface at the centres of the faces of the cube, and the figures which can thereafter be superposed upon the cube are only those similar to the fully completed constructure.

The octahedra thus invited to be interposed in the circuits, being considered as in fact interposed, the entire figure is an octahedron of the second order, on the faces of which tetrahedra of three times the edge of the original superposed tetrahedra may be superposed and considered as the analogues of wider processes of reasoning, and the building up of the figure and the processes of reasoning may be in like manner continued indefinitely, widening as they progress throughout all conceivable regions of space.

The author closes (as both logically and geometrically he should, returning to the point of beginning) by recurring to the main question concerning which it has just occurred to him that it has nowhere throughout the book been formulated as an interrogatory, and that if so formulated, it could be enlarged and put forth as follows:

Would the processes of geometry be in any wise affected by changing the forms of the units of measure of surface and solidity, and if yea, then how, in respect to each possible change to regular figures, favorably or unfavorably?

And it seems to him that put in this form it carries with it conviction of the necessity of an affirmative answer to the main question as herein first stated, as it presents immediately the antithesis of change from the square and cube, both regular figures, to the regular triangle and tetrahedron on the one side, or to the regular pentagon and dodecahedron on the other.

In medio tutissimus ibis. The broad and beaten middle highway is undoubtedly the safest for the multitude, who can thereby easily, although very indirectly, ascend the hill of science to the table-lands on which they are content to dwell, but for the expert climber who, as an explorer, would reach, or, as a guide, would lead others to the summit, the shorter, narrower, more nearly direct and capable of being reduced to absolutely direct path pointed out by nature, but hitherto wholly untracked, is, as it would seem to the author, far better adapted, and if it had been sought would have been readily found and, as it would also seem, unquestionably pursued.

Whether or not in these days of marvellous progress, this path in its zigzag course or made straight by span and trestle, viaduct, cut, and tunnel, will be opened as a new highway, remains to be seen, but the author verily believes that if not in these, it will be in later days.

QUESTIONS IN MATHEMATICS.

POSTSCRIPT.

Strike out, beginning with the 3d ¶ from the bottom of page 70 to end of 7th line of page 107, and insert instead:

The greatest second and third powers contained in any given number and the roots thereof respectively may be found by considering the given number as the area of a regular triangle with reference to the second power and as the volume of a regular tetrahedron with reference to the third, and analyzing those figures to find the side of the triangle and the edge of the tetrahedron, which are the second and third roots respectively.

Let it be required to find the side of a regular triangle the area of which is given as 127449.

The figure consists of as many rows of regular triangles of side 1 (including the initial triangle considered as a row) as the number of the required side, each row after the first exceeding the next preceding row by 2.

There are three places of figures in the number of the side, the first figure being 3, to which affix two ciphers, making 300.

From the given number	127449
subtract the number of triangles contained in the first 300 rows = 300^2 = ...	90000
leaving remainder	37449

The difference between the numbers of triangles contained in the 300th and 310th rows = (300 + 310) × 10 = 6100, and between those contained in the 310th and 320th rows = (310 + 320) × 10 = 6300, and so on, each successive difference exceeding the next preceding by 200. Omitting the ciphers, the first five differences and their sum are 61 + 63 + 65 + 67 + 69 = 60 × 5 + 5^2 = ... 325

which subtracted from the first three figures of the remainder, as above, leaves second remainder 4949

5, being the number of differences contained in the first remainder is thus found to be the second figure of the side.

The difference between the numbers of triangles contained in the 350th and 351st rows = 350 + 351 = 701. It will now be readily seen that 7 is the third and last figure of the side.

700 × 7 + (1 + 3 + 5 + 7 + 9 + 11 + 13 = 7^2) = ... 4949

which subtract from second remainder. . . . . .

There being no third remainder, the given number is a perfect second power, the root of which is 357.

Let the given number be 23109986.

There are four places of figures in the root. First figure 4, to which affix three ciphers.

Given number, ...	23109986
$-\ 4000^2 =$	16000000
$=$	7109986
$40 + 41 = 81.$	
$80 \times 8 + 8^2 =$	704
which subtracted from first three figures of remainder as above leaves second remainder,	69986

8 is thus found to be the second figure of the root.

$480 + 481 = 961$, which exceeds the first three figures of the remainder.

The third figure of the root is thereby found to be 0.

$4800 + 4801 = 9601.$	
$9600 \times 7 + 7^2 =$	67249

7 is thus found to be the fourth and last figure of the root, making the whole root 4807.

Sum last found subtracted from remainder, as before, leaves remainder of given number over and above the greatest second power therein contained, ...	2737
which subtracted from given number, . .	23109986
gives such greatest second power, $4807^2 =$.	23107249

The volume in tetrahedra of edge 1 of a regular tetrahedron of any given edge is equal to the product of the number of octahedra of edge 1 contained in such regular tetrahedron multiplied by 6 + the number of the edge. Conversely, the number of octahedra of edge 1 contained in a regular tetrahedron of any given volume in tetrahedra of edge 1 is equal to the quotient of such volume divided by 6 — the quotient of the edge of such regular tetrahedron divided by 6.

Let it be required to find the edge of a regular tetrahedron the volume of which is given as 1367631.

There are three places of figures in the edge, and the first figure of the edge is 1, to which affix two ciphers.

Find the quotient by 6 of the given volume. $\frac{1367631}{6} =$ 227938.5

The number of octahedra of edge 1 contained in a regular tetrahedron of edge 100 is equal to $100^2 \times \frac{100}{6} - \frac{100}{6} =$ . . . 166650

which, subtracted from the quotient as above, leaves remainder 61288.5

The difference between the numbers of octahedra of edge 1 contained in two regular tetrahedra of edges 100 and 110 respectively $= \frac{11 \times 10}{2}$ (= 55) with three ciphers affixed + the number of such octahedra in a regular tetrahedron of edge 10 ($= 10^2 \times \frac{10}{6} - \frac{10}{6} = 165$) = ... 55165

which, subtracted from the remainder as above, leaves second remainder 6123.5

The like difference in respect to two regular tetrahedra of edges 110 and 111 respectively is equal to the number of octahedra contained in course 111 of the latter $= \frac{111 \times 110}{2}$ (see last ¶ on page 68)= . . 6105

which subtracted from the second remainder as above leaves third remainder . . . 18.5

$18.5 \times 6 = 111$, which is the required edge.

VERIFICATION.

The number of octahedra of edge 1 contained in the regular tetrahedron which has been thus analyzed consists as follows:

Of the number in 100	166650
+ the difference between 100 and 110 . .	55165
+ " " " 110 " 111 . .	6105
Total number of octahedra . . .	227920
To find their volume in tetrahedra of edge 1, multiply the number by	4
Volume of the octahedra	911680
To find the number of tetrahedra of edge 1 contained in such regular tetrahedron, multiply the number of the octahedra by 2 and to the product add the number of the edge 111 (see 6th ¶ on page 69) = ...	455951
111^3 = ...	1367631

This, as will be readily seen, is equivalent to multiplying the number of octahedra by 6 and adding to the product the number of the edge.

Let the given number, the greatest third power and remainder over, if any, contained in which are required, and also the root of the power, be 111111111111.

No. of places of figs. in root 4.

First fig. of root 4, to which affix three ciphers, making 4000.

Quotient by 6 of given no. = . . .	18518518518.5
No. oct. in 4000 = $4000^2 \times \frac{4000}{6} - \frac{4000}{6}$ = ...	10666666000
which subtracted from quo., as above, leaves remainder	7851852518.5

To find the second figure of the root.

Diff. bet. nos. oct. in 4000 and 4100

$= \frac{41 \times 40}{2}$ (= 820) with 6 ciphers affixed

+ no. oct. in 100 (166650) = 820166650

+ diff. bet. 4100 and 4200

$= \frac{42 \times 41}{2}$ (= 820 + 41 = 861)

with 6 ciphers affixed + no.

oct. in 100, as before = 861166650

+ diff. bet. 4200 and 4300

= 861 + 42, etc., as before = 903166650

+ diff. bet. 4300 and 4400

= 903 + 43, etc., as before = 946166650

+ diff. bet. 4400 and 4500

= 946 + 44, etc., as before = 990166650

+ diff. bet. 4500 and 4600

= 990 + 45, etc., as before = 1035166650

+ diff. bet. 4600 and 4700

= 1035 + 46, etc., as before = 1081166650

+ diff. bet 4700 and 4800

= 1081 + 47, etc., as before = 1128166650 = 7765333200

= diff. bet. nos. oct. in 4000 and 4800,
which, subtracted from rem. as above,
leaves second rem. 86519318.5

8, being the number of the differences thus added together, is the second figure of the root. Substitute same for the first cipher affixed, making 4800.

To find the third figure of the root.

Diff. bet. nos. oct. in 4800 and 4810

$= \frac{481 \times 480}{2}$ (= 115440) with 3 ciphers

affixed + no. oct. in 10 (165) = 115440165

This difference being in excess of the remainder, the third figure of the root is thereby found to be 0. Retain second cipher affixed, making still 4800.

To find the fourth and last figure of the root.

Diff. bet. nos. oct. in 4800 and 4801

$= \frac{4801 \times 4800}{2}$	=	11522400		
+ 11522400 + 4801	=	11527201		
+ 11527201 + 4802	=	11532003		
+ 11532003 + 4803	=	11536806		
+ 11536806 + 4804	=	11541610		
+ 11541610 + 4805	=	11546415		
+ 11546415 + 4806	=	11551221	=	80757656

= diff. bet. nos. oct. in 4800 and 4807, which, subtracted from rem. as above, leaves rem. 5761662.5

7, being the number of the differences thus added together, is the fourth and last figure of the root, and being substituted for the last cipher affixed, makes the whole root 4807.

Subtract quo. by 6 of whole root from preceding remainder. $\frac{4807}{6} =$ . . 801.166

Last remainder 5760861.333

which restored to volume by multiplication by 6

= remainder of given number over greatest third power therein contained . 34565168

Subtract same from given number 111111111111

Rem. = greatest third power required, 111076545943

The foregoing example exhibits the process in its full elaboration. But it may be very considerably shortened as follows :

1st. In respect to finding no. oct. in 4000.

There is no octahedron, except fractional, in a regular tetrahedron of edge 1. $1^2 \times \frac{1}{6} - \frac{1}{6} = 0$.

No. oct. in 10 $= 10^2 \times \frac{10}{6} - \frac{10}{6} = 165$.

No. oct. in 100 = no. oct. in 10 with 3 ciphers affixed = . . . 165000

\+ no oct. in 10 × 10 = 1650 = 166650.

No. oct. in 1000 = no. oct. in 10 with 6 ciphers affixed = . . . 165000000

\+ no. oct. in 100 × 10 = 1666500 = 166666500

Thus it will be seen that the number of octahedra contained in every regular tetrahedron, the first figure of the edge of which is 1 and is followed by ciphers throughout, may be found by inserting for each cipher *beyond the first* two 6's between 6 and 5 as found in 10, and affixing for each such cipher one cipher.

Thus no. oct. in 10000 = 166666665000.

The number of octahedra contained in any regular tetrahedron, the first figure of the edge of which is any one of the other eight digits and is followed by ciphers throughout, is equal to the number in a regular tetrahedron of which the first figure is the edge with three ciphers affixed thereto for each cipher in the edge (but observe, *including and not beyond the first cipher*, as before) + the number in 10, 100, 1000, and so on (the ciphers being the same in number as those in the edge) × the first figure of the edge.

Thus no. oct. in 20 $= 2^2 \times \frac{2}{6} - \frac{2}{6} = 1$ which with 3 ciphers affixed = ... 1000
+ no. oct. in 10 (165) $\times 2 =$ 330

$= 20^2 \times \frac{20}{6} - \frac{20}{6} =$ 1330

No. oct. in 300 $= 3^2 \times \frac{3}{6} - \frac{3}{6} = 4$ which with 6 ciphers affixed = ... 4000000
+ no. oct. in 100 (166650) $\times 3 =$ . . . 499950

$= 300^2 \times \frac{300}{6} - \frac{300}{6} =$ 4499950

No. oct. in 4000 $= 4^2 \times \frac{4}{6} - \frac{4}{6} = 10$ which with 9 ciphers affixed = ... 10000000000
+ no. oct. in 1000 (166666500) $\times 4 =$ . . 666666000

$= 4000^2 \times \frac{4000}{6} - \frac{4000}{6} =$ 10666666000

as in the example.

2d. In respect to finding the second figure of the root (8).

The first remainder in the example is 7851852518.5 and the first difference is 820166650.

The second figure of the root may be found by means of the first four figures of the remainder and the first three figures of the difference in connection with the following series :

Nos. of terms.	1	2	3	4	5	6	7	8	9
Series.	0.	1.	3.	6.	10.	15.	21.	28.	36

Find the *greatest* term of the series, the product of 40 multiplied by which will, when increased by the sum of all the terms up to and including the term and added to the product of the first three figures of the difference multiplied by the number of the term, yield a sum less than the first four figures of the remainder.

40 × 28 = ...	1120
+ sum of all the terms up to and including 28 =	84
+ 820 × 8 = ...	6560
=	7764

which is less than 7851, the first four figures of the remainder. If 36, the 9th term of the series, had been taken, the resulting sum would have been 8940, which is greater than 7851. 28 is, therefore, the *greatest* term required, and the number thereof, 8, is the second figure of the root.

The rationale of this process is as follows:

By reference to the example it will be seen that the first three figures of the second difference exceed those of the first difference by 41; that those of the third difference exceed those of the second by 42, and so on up to the eighth difference in which the excess over the seventh is 47. 40 is contained in the sum of these successive increments 30 times, but 2 of these times (the excess over 28) constitute part of that part of the sum of the increments which results from the addition together of the products of the units 1, 2, 3, etc., up to 7 multiplied by 7, 6, 5, etc., down to 1 respectively = 84.

The series consists of the numbers of octahedra contained in the several courses of a regular tetrahedron of edge 9, and the sum of the terms up to and including any term is the number of octahedra contained in a regular tetrahedron of which the number of the last included term is the edge.

3d. The next step of the process is to find the difference between the numbers of octahedra contained in two regular tetrahedra of edges 4000 and 4800 (root so far as found) respectively, which may be done as follows :

Diff. bet. nos. oct. in 4000 and 4100 as before found = ...	820166650
× second figure of root as found . .	8
=	6561333200
to which add 40 × 28 (8th term of series) = 1120 with 6 ciphers affixed = . .	1120000000
+ sum of series up to and including 8th term = 84 with 6 ciphers affixed = . .	84000000
=	7765333200

= diff. bet. nos. oct. in 4000 and 4800 as found in the example by the finding and addition together of the eight successive differences between 4000 and 4100, 4100 and 4200, and so on up to 4700 and 4800.

But the remainder and first difference, as before found, may be such that it will at once be perceived what the required figure of the root is without recourse to the series, and in such case the required sum of all the differences may be found by a shorter process, which in this example would be as follows :

Diff. bet. nos. oct. in 4000 and 4800 = $\frac{48 \times 40}{2}$ =

960 × 8 = 7680, which with 6 ciphers affixed = ...		7680000000
+ no. oct. in 800 = no. oct. in 8 with 6 ciphers affixed = . . .	84000000	
+ no in 100 (166650) × 8 =	1333200 =	85333200
=		7765333200

as before and as in the example.

4th. In respect to finding the fourth and last figure of the root (7) and at the same time the difference between nos. oct. in 4800 and 4807.

The remainder shown in the example is 86519318.5, and the first difference is 11522400

and by comparison and consideration of them in connection with the series, it will be found that term 21 of the latter is the required term, and that the number thereof (7) is the fourth and last figure of the root.

Diff. as above × no. of required term	7
=	80656800
+ 4800 (as above) × 21, required term, =	100800
+ sum of series up to and including term 21 = ...	56
=	80757656

= diff. bet. nos. oct. in 4800 and 4807 as found in the example.

Or, the last figure of the root being perceived by comparison of the remainder and first difference found as above to be 7, the whole difference may be found by the shorter and direct process before shown, as follows:

$$\text{Diff. bet. nos. oct. in 4800 and 4807} = \frac{4807 \times 4800}{2}$$

=	11536800
×	7
=	80757600
+ no. oct. in 7 = ...	56
=	80757656

Let the given number be 45499294.

There are three places of figures in the root, and the first figure of the root is 3.

$\frac{45499294}{6} =$		7583215.666
$- 3^2 \times \frac{3}{6} - \frac{3}{6}$ with 6 ciphers affixed		
$+ 166650 \times 3 =$		4499950.
=		3083265.666
$- \frac{31 \times 30}{2}$ with 3 ciphers affixed		
$+ 165 =$	465165	
$\times$ no. of term 10 of series, such no. being the second figure of the root,	5	
=	2325825	
$+ 30 \times 10$ with 3 ciphers affixed =	300000	
$+ 0 + 1 + 3 + 6 + 10$ with 3 ciphers affixed = . . .	20000 =	2645825
=		437440.666
$- \frac{351 \times 350}{2} =$	61425	
$\times$ no. of term 21 of series, such no. being the third and last figure of the root	7	
=	429975	
$+ 350 \times 21 =$	7350	
$+ 0 + 1 + 3 + 6 + 10 + 15 + 21 =$	56 =	437381
=		59.666
− quo. by 6 of root as found, $\frac{357}{6} =$		59.5
=		0.166
$\times$		6
= rem. of given no. over greatest third power therein contained .		1
which subtracted from given no. .		45499294
gives such greatest third power.		
$357^3 =$		45499293

In the usual arithmetical process based upon an analysis of the cube each step after the first is in the first instance tentative; but in the process based upon an analysis of the regular tetrahedron there is no tentative step, but absolute certainty throughout.

Which of the two plane figures, the regular triangle or the square, and of the two solid figures, the regular tetrahedron or the cube, respectively, seems the better adapted to finding by an analysis thereof the greatest second or third power, respectively, the root thereof and the remainder over, if any, contained in the given area or volume of any figure, plane or solid, perfect or imperfect?